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# STABILITY OF ATTITUDE CONTROL SYSTEMS ACTED UPON BY RANDOM PERTURBATIONS

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## PREFACE

This report contains the results of the first phase of a study of the stability of attitude control systems acted upon by random perturbations, performed from July 1965 to June 1966 at the General Precision Aerospace Research Center under Contract NAS 12-48 with the Electronics Research Center, National Aeronautics and Space Administration.

The principal investigator was Dr. Bernard Friedland; contributors included Professor Philip E. Sarachik (New York University), Dr. Frederick E. Thau, Mr. Jordan Ellis and Mr. Victor D. Cohen. Mr. William Cashman of the Electronics Research Center served as NASA Technical Monitor. Professor W. M. Wonham of Brown University provided a number of most fruitful discussions.

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# 1. INTRODUCTION

Over the past few years, dynamic processes with random excitations have been the subject of increasing attention by investigators in the field of modern control theory. The growth of interest in these problems can be attributed, in part, to the fact that many of the basic theoretical problems of deterministic control theory are by now fairly well in hand. The "easy" problems have been solved, and the difficulties inherent in the "hard" problems are widely recognized. Although practical application of the results of modern deterministic control theory is still not too common, the opportunities for fundamental theoretical contributions appear to be greater in the area of stochastic control theory.

Random disturbances and changes in the process parameters (both of which can be loosely interpreted as excitations) are always a practical problem in control system design. They are, in fact, the reason underlying the very use of feedback control. Therefore, the results of stochastic control theory can be expected to have significant impact on practical control system designs of the future, particularly in space applications where lifetime and extreme precision of performance are at a premium.

The principal distinction between current approaches to randomly-excited control systems and the approach used by engineers in the period from about 1945 to 1960 is that the current approaches make direct use of the differential equations governing the behavior of the process. The earlier approach is concerned primarily with frequency-domain representations (spectral density, etc.) and is consequently confined largely to the steady-state behavior of linear, time-invariant systems excited by stationary white noise. The current approaches appear to be a natural extension of the deterministic state-space techniques to systems with random excitations and liberate the study of randomly-excited dynamic systems from the constraints imposed by the methods of frequency domain analysis.

The evolution of probabilities in state space is the general problem of interest. As we shall endeavor to show subsequently, this evolution is governed by partial differential equations, the nature of which depends on both the dynamic process under investigation and the characteristics of the random excitation to the process. Although the approach to the study of randomly-excited dynamic processes through the use of the equations governing the evolution of probabilities is a relatively recent development in control theory, this approach has been used by physicists and mathematical statisticians since the early part of the century. As early as 1905, Einstein [1] inferred from basic physical considerations that the random motion of particles suspended in a fluid can be described by the diffusion equation (the same partial differential equation that governs the conduction of heat in a solid). This motion was first described by Robert Brown in 1826 and is now called Brownian motion. Generalization of Einstein's results occupied a number of distinguished theoretical physicists during the early part of the century. So many investigators were involved in these studies, that some of the fundamental equations of the theory are identified by the names of different persons (much to the confusion of workers in the field today. For example, the "Chapman-Kolmogorov equation", referred to in section 3, is often known as the "Smoluchowski equation".) The status of the theory from the physicists's viewpoint, as it had evolved from Einstein's work to the early 1940's, was reviewed in a monograph-length paper by Chandrasekhar*[2] and in a paper by Wang and Uhlenbeck*[3].

Mathematical rigor and elegance were introduced to the theory of stochastic processes beginning about 1930. The names of Wiener, Lévy, Kolmogorov, Feller, Doob, Itô, and Dynkin are prominently associated with this work. The two-volume book by Dynkin [4] appears to be the most elegant and comprehensive (if not readily comprehensible) treatment to date.

The main lines of investigation in modern deterministic control theory are stability theory and optimum control theory. Similarly, these lines of investigation are being pursued in connection with stochastic control theory. Our attention will be restricted to the problems associated with stability.

Early work on stochastic stability was done by Bertram and Sarachik [5] and by Kats and Krasovskii [6]. These investigations were concerned with "stability in moment" and made use of a stochastic version of Lyapunov's second method. The practical deficiency of "stability in moment" was pointed out by Kushner[7], who, along with Khasminskii [8], considered "almost-certain stability", or "stability with probability one", which is a more meaningful form of stability. Stochastic Lyapunov theory was also employed in these investigations. Almost-certain stability is so strong a property that it is not possessed even by stable deterministic systems when the random excitation is constantly-acting white noise. Kushner, through the use of stochastic Lyapunov Theory [9], and Wonham, by means of properties of differential operators [10] have been considering problems associated with estimating lifetime in systems which are not stable with probability one.

* Six classic papers in the field are reprinted in N. Wax (ed.), Selected Papers on Noise and Stochastic Processes, Dover Publications, New York, N.Y., 1954. This book includes the papers by Chandrasekhar and by Wang and Uhlenbeck.

The objectives of the investigation reported herein were as follows:

1. Organization of existing results and interpretation of these results in a framework which is intuitively appealing.

2. Development of appropriate models for the sources of random disturbances acting upon space vehicles.

3. Extension, where, possible of existing techniques, to the solution of significant problems.

4. Studies of typical control problems.

5. Identification of problems for further study.

It was felt that in making the existing theory more accessible to engineers it would be desirable to derive the fundamental equations ("forward" and "backward" equation) for the evolution of Markov processes with the minimum or mathematical sophistication. These derivations are contained in Sections 3 and 4. Although the results obtained are formal, they can be accepted with confidence, since rigorous developments are available [4].

The starting point for models of disturbance sources is the "Poisson impulse process" described in Section 2. The integral of this process, as the number of impulses per unit time becomes infinite and the areas of the impulses become infinitesimal, is the "Wiener process" which is the basis of most analytical investigations. Mathematical properties of these processes are discussed in Section 2, and estimates of the parameters of such processes for various physical disturbance sources are obtained in Section 8.

"Confinement probability", i.e., the probability of not leaving a given region in state space for a specified time interval starting at a point in this region, is considered in Section 5 as the most natural extension of deterministic stability; we also believe that confinement probability is an important consideration in design of space vehicle control systems. Analytical and numerical techniques for estimation of confinement probability, and the "half-life" and "mean confinement time", which are related thereto, are considered in Section 6. These techniques are used in Section 7 to perform some calculations on several first- and second-order linear and nonlinear systems. Some of the implications of these calculations as regards a typical space vehicle are discussed in Section 8.

Problems for further investigation, which include that of obtaining better estimates of the statistical parameters of the random excitations, and of finding better analytical techniques for estimation of lifetime, are summarized in Section 9.

Two appendices are included. Appendix 1 is an annotated bibliography, and Appendix 2 is a description of the computer program which was used for the Monte-Carlo simulations described herein.

## 2. MODELS OF RANDOM EXCITATION

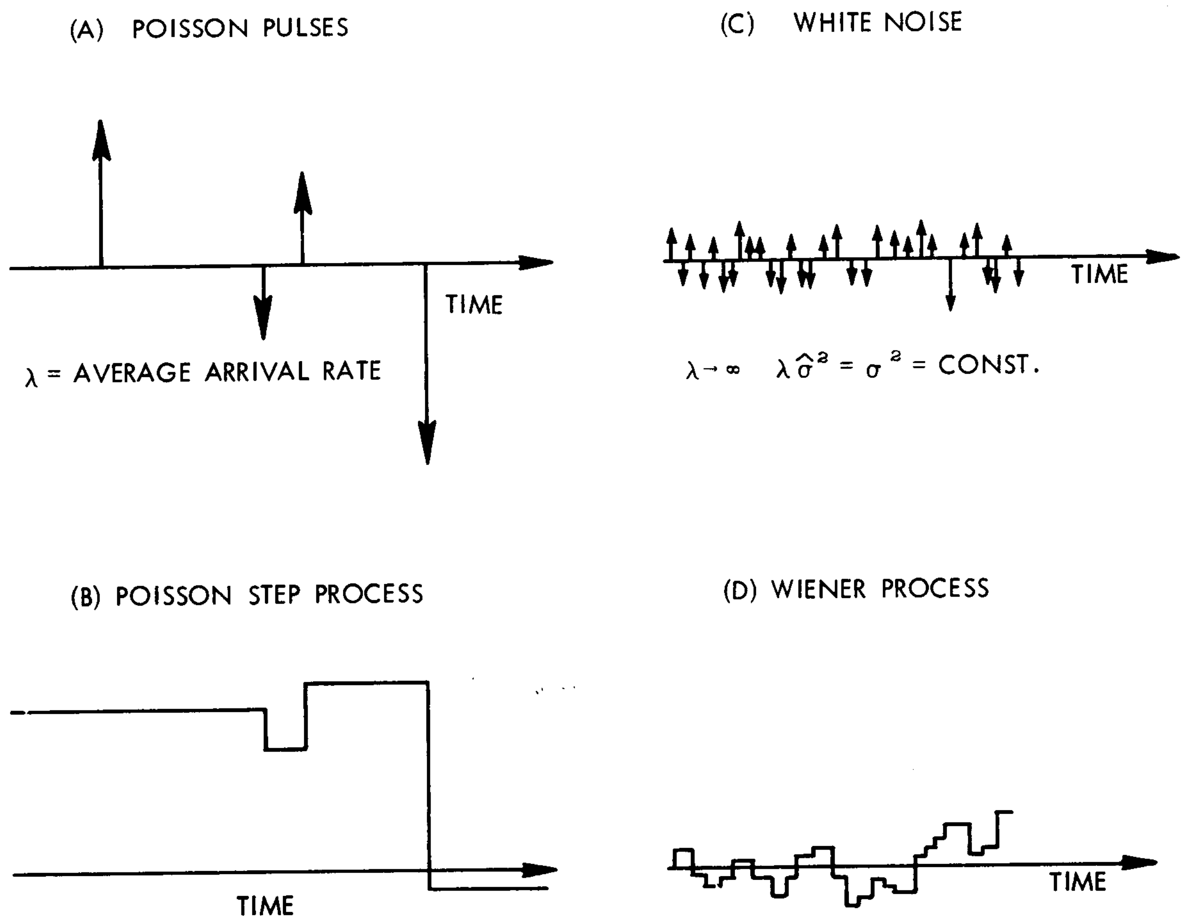
One of the major purposes of a control system is to maintain motion of the controlled object within acceptable limits in the presence of random disturbances which cause the object to deviate from the motion desired. In the case of space vehicles, the random disturbances can be caused by external phenomena such as micrometeoroid bombardment or by internal phenomena such as the motion of the vehicle occupants or the imprecise operation of the control system itself.

An analytical investigation of the behavior of an actual controlled space vehicle in the presence of random disturbances entails the establishment of analytical models for the random disturbance processes. These models must be simple enough to permit analytical calculations; at the same time they must adequately represent the physical phenomena. The selection of models to meet these requirements is a major problem which requires much more attention than it has heretofore received. In this investigation we have attempted to characterize several of the obvious sources of random disturbances and to estimate the statistical parameters associated with these sources. The discussion of our results is contained in Section 8 in which the models used are based on the ideal models considered in the present section.

A random process consisting of impulses of random area (or "strength") and random arrival times, as depicted in Figure 2-1a has intuitive appeal to the engineer who is accustomed to dealing with deterministic linear systems. The response of such systems is obtained by superposition of the impulse responses. For mathematical reasons, and also because the effect of the impulses is related to the total area of the impulses, it is more convenient to deal with the integral of the impulsive process of Figure 2-1a as depicted in Figure 2-1b. Consider now a time interval of duration dt during which any number of transitions may occur. The probability that the total change in height due to these transitions lies between z and z + dz is given by

$$p(z\,,\,dt)\;dz = \sum_{n=0}^{\infty} P_n\,(dt)\,\gamma_n\,(z)\;dz \qquad (2.1)$$

## FIGURE 2-1
## DISTURBANCE SOURCES



where $P_n(dt)$ is the probability that exactly n transitions occur in the interval dt and $\gamma_n(z)\,dz$ is the probability that the level change due to exactly n transitions lies between z and z + dz. In most situations it is reasonable to assume that the probability of occurrence of exactly n transitions in the interval dt has a Poisson distribution, i.e.,

$$P_n(dt) = e^{-\lambda dt}(\lambda\ dt)^n/n! \tag{2.2}$$

where $\lambda$ is the "average occurrence rate." Hence, (2.1) becomes

$$p(z\ ,\ dt)\,dz = e^{-\lambda dt} \sum_{n=0}^{\infty} \frac{(\lambda\ dt)^n}{n!}\ \gamma_n(z)\ dz \tag{2.3}$$

If the height at the beginning of the interval dt is specified as y, then, given this information, the probability $p(x\ ,\ dt|y)\,dx$ that the height at the end of the interval lies between x and x + dx is equal to the probability $p(z\ ,\ dt)\,dz$ that the change lies between z and z + dz when $x = z + y$. Thus, from (2.3),

$$P(x\ ,\ dt|y) = e^{-\lambda dt} \sum_{n=0}^{\infty} \frac{(\lambda\ dt)^n}{n!}\ \gamma_n(x - y) \tag{2.4}$$

A process with density (2.4) is called a Poisson step process. The original process of Figure 2-1a which is integrated to produce the Poisson step process may be referred to as a Poisson impulse process.

Let $\gamma(z)\,dz$ be the probability that the height of a single step is between z and z + dz. Then, provided the heights of the individual steps are independent, we have

$$\underbrace{\gamma_n(z) = \gamma(z) * \ldots * \gamma(z),}_{\text{n-fold convolution}} \qquad \gamma_0(z) = \delta(z)\ \ \text{(delta function)} \tag{2.5}$$

To avoid dealing with n-fold convolutions it is convenient to use the characteristic function $C(\nu\ ,\ dt|y)$.

$$C(\nu\ ,\ dt|y) = \int_{-\infty}^{+\infty} e^{j\nu x}\ p(x\ ,\ dt|y)\ dx$$

$$= e^{-\lambda dt + j\nu y} \sum_{n=0}^{\infty} \frac{[\lambda\, dt\, \Gamma(\nu)]^n}{n!} \tag{2.6}$$

$$= \exp[-\lambda dt (1 - \Gamma(\nu)) + j\nu y]$$

where

$$\Gamma(\nu) = \int_{-\infty}^{+\infty} e^{j\nu z} \gamma(z)\, dz \qquad j = \sqrt{-1}$$

is the characteristic function of the probability density function of the area of one impulse. The characteristic function (2.6) can be written

$$C(\nu, dt \mid y) = \{\exp[\Gamma(\nu) - 1 + \frac{j\nu y}{\Lambda}]\}^{\Lambda}, \quad \Lambda = \lambda\, dt \tag{2.7}$$

where $\Lambda$ is the average number of impulses in the interval $dt$. Hence it is seen that $C(\nu, dt \mid y)$ is the characteristic function for the sum of $\Lambda$ identically-distributed random variables, each of which has the characteristic function

$$B(\nu) = \exp[\Gamma(\nu) - 1 + j(\nu y/\Lambda)] \tag{2.8}$$

By virtue of the central-limit theorem, as $\Lambda \to \infty$, $C(\nu, dt \mid y)$ approaches the characteristic function of a Gaussian random variable with mean $\Lambda\mu$ and variance $\Lambda\hat{\sigma}^2$, where $\mu$ and $\hat{\sigma}^2$ are the mean and variance, respectively, of the density function corresponding to $B(\nu)$:

$$\mu = -j \left.\frac{dB(\nu)}{d\nu}\right|_{\nu=0} = \frac{y}{\Lambda}$$

(when $\gamma(x)$ has zero mean)

$$\hat{\sigma}^2 = -\left.\frac{d^2 B(\nu)}{d\nu^2}\right|_{\nu=0} \tag{2.9}$$

Consequently

$$C(\nu, dt \mid y) \to \exp\left(j\nu y - \frac{\Lambda\hat{\sigma}^2\nu^2}{2}\right) = \exp\left(j\nu y - \frac{\sigma^2\, dt\, \nu^2}{2}\right)$$

where $\sigma^2 = \lambda\hat{\sigma}^2$. The corresponding probability density function $p(x, dt \mid y)$ is given by

$$p(x\,,\,dt|y) = \frac{1}{\sqrt{2\pi\ dt}\ \sigma} \exp\left(-\frac{(x-y)^2}{2\sigma^2\ dt}\right) \tag{2.10}$$

where $p(x\,,\,dt|y)\,dx$ is the probability that the height at time $t + dt$ lies between $x$ and $x + dx$, given that the height at time $t$ was $y$.

A process having the probability density function (2.10) is called a Wiener process. Thus, a Wiener process can be regarded as the limiting form of a Poisson step process as the average transition rate becomes infinite and the variance of the transitions approaches zero in such a manner that $\lambda\hat{\sigma}^2 = \sigma^2 = \text{const}$, as visualized in Figure 2-1d.

The Wiener process is thus visualized as the result of integrating a Poisson process and then letting the transition rate become infinite. An alternate visualization of the Wiener process is obtained by first letting the occurrence rate of the impulses becoming infinite (Figure 2-1c) and then integrating the result. The random process obtained prior to the integration is a visualization of white noise, the formal derivative of the Wiener process.

In classical engineering approaches, equation processes are characterized by either autocorrelation functions $\varphi_{xx}(t)$ or spectral densities $S_{xx}(\omega)$. It has been shown [11] that the "spectral density"* of the Poisson step process (2.4) is

$$S_{xx}(\omega) = \frac{\lambda\hat{\sigma}^2}{\omega^2} = \frac{\sigma^2}{\omega^2} \tag{2.11}$$

Thus, we see that as $\lambda \to \infty$ and $\lambda\hat{\sigma}^2 = \sigma^2 = \text{constant}$ the spectral density (2.11) does not change. Thus, (2.11) also gives the spectral density of the Wiener process (2.10). Since the Poisson step and Wiener process result from integrating Poisson impulse and white noises processes respectively, we find that these "derivative" processes both have zero mean and spectral densities

$$S_{xx}(\omega) = \sigma^2 \tag{2.12}$$

or autocorrelation functions

$$\varphi_{xx}(t-\tau) = \sigma^2\delta(t-\tau) \tag{2.13}$$

We see that knowledge of only spectral densities is not sufficient to distinguish between a white noise and a Poisson impulse process or between a Wiener and a Poisson step process.

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* This is formal, since the process is not stationary.

For reasons such as this we see that the classical engineering approach to the study of random processes is not adequate in problems where distinctions between the underlying random processes are of importance.

The Wiener process has properties which make it convenient to deal with in detailed analytical calculations, and it thus serves as the source process in most theoretical studies. Since in a physical process, however, there is always a finite upper limit on the number of occurrences in any time interval, the Wiener process should be regarded as a convenient, but unattainable abstraction. In situations in which the Wiener process or white noise, its derivative, lead to results which do not appear consistent with reality, it would be well to examine whether the difficulty can be avoided by the use of a more realistic model of the source process.

# 3. EVOLUTION OF MARKOV PROCESSES

## 3.1 Chapman-Kolmogorov Equations

The basic problem in the study of stochastic processes is the mathematical description of the manner in which characteristics of a process evolve in state space. As a first step towards obtaining this description, consider a vector random process $\{X(t)\}$ with components at time $t$ being the random variables $X_i(t)$, $i = 1,2, \ldots, n$; $X(t)$ thus denotes a random point of an n-dimensional Euclidean space. We will denote a possible value of the random component $X_i$ at time $t^j$ by $x_i^j$. Any random process can be characterized by its joint probability densities of all order $k$, $f_k(x^1, t^1; \ldots; x^k, t^k)$, where $f_k(x^1, t^1; \ldots; x^k, t^k)\, dx^1\, dx^2 \ldots dx^k$ denotes the joint probability of finding each component $X_i$ in the range $(x_i^1, x_i^1 + dx_i^1)$ at time $t^1$, and in the range $(x_i^2, x_i^2 + dx_i^2)$ at time $t^2, \ldots$, and in the range $(x_i^k, x_i^k + dx_i^k)$ at time $t^k$.

A particular random process which may be used as a mathematical model for a large class of physical processes is the Markov process which is defined by the additional property

$$f(x^k, t^k | x^{k-1}, t^{k-1}; x^{k-2}, t^{k-2}; \ldots; x^1, t^1) = f(x^k, t^k | x^{k-1}, t^{k-1}) \qquad \text{for } t^1 \leq t^2 \leq \ldots \leq t^k \tag{3.1}$$

That is to say, for such processes the conditional probability that $X(t^k)$ be in a neighborhood of the point $x^k$ given that $X(t^i) = x^i$ (for all $i = 1, \ldots, k-1$) is the same as the conditional probability given only that $X(t^{k-1}) = x^{k-1}$. The expression $f(x^k, t^k | x^{k-1}, t^{k-1})$ is called the transition probability density of the Markov process.

Since, in general,

$$f_k(x^1, t^1; \ldots; x^k, t^k) = f(x^k, t^k | x^{k-1}, t^{k-1}; \ldots; x^1, t^1)\, f_{k-1}(x^1, t^1; \ldots; x^{k-1}, t^{k-1})$$

we find that for Markov processes, all finite order densities are given by

$$f_k(x^1, t^1; \ldots; x^k, t^k) = f(x^k, t^k | x^{k-1}, t^{k-1})\, f_{k-1}(x^1, t^1; \ldots; x^{k-1}, t^{k-1})$$

$$= f(x^k, t^k | x^{k-1}, t^{k-1})\, f(x^{k-1}, t^{k-1} | x^{k-2}, t^{k-2}) \cdot \ldots \cdot f(x^2, t^2 | x^1, t^1)\, f_1(x^1, t^1)$$

Thus for a Markov process, all finite order joint densities are specified by the first order density $f_1$ and the transition probability density. In the remainder of the report we will write the transition density (3.1) as $f(x, t|y, \tau)$ where $x$ denotes the value of $X$ at the present time $t$ and $y$ denotes the value of $X$ at the initial time $\tau$.

For a Markov process with $\tau \le s \le t$ we have from the above expression

$$f_3(y, \tau; \xi, s; x, t) = f(x, t|\xi, s)\, f(\xi, s|y, \tau)\, f_1(y, \tau) \tag{3.2}$$

and

$$f_2(y, \tau; x, t) = f(x, t|y, \tau)\, f_1(y, \tau) \tag{3.3}$$

Moreover, joint densities, by definition, satisfy

$$f_2(y, \tau; x, t) = \int f_3(y, \tau; \xi, s; x, t)\, d\xi \tag{3.4}$$

where $\int$ denotes integration over the entire n-dimensional Euclidean space. Substitution of (3.2) and (3.3) into (3.4) yields

$$f(x, t|y, \tau) = \int f(x, t|\xi, s)\, f(\xi, s|y, \tau)\, d\xi \tag{3.5}$$

which is called the <u>Chapman-Kolmogorov equation for transition densities</u> and is a basic equation in the study of Markov processes. It is important to note that although for a general random process the right hand side of (3.5) depends on $s$, for a Markov process it is independent of $s \epsilon [\tau, t]$.

A relation similar to (3.5) may be obtained for transition distributions, defined by

$$F(x, t|y, \tau) = \int^{x} f(\xi, t|y, \tau)\, d\xi \tag{3.6}$$

where $\int^{x}$ means $\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n}$

Upon integrating both sides of (3.5) with respect to $x$, interchanging the order of integration on the right-hand side, we obtain

$$F(x, t|y, \tau) = \int F(x, t|\xi, s)\, f(\xi, s|y, \tau)\, d\xi \tag{3.7}$$

which is the <u>Chapman-Kolmogorov equation for transition distributions</u>.

A Markov process is said to be stationary or to have a stationary probability density if

$$f(x, t|y . \tau) = f(x, t-\tau|y, 0) \equiv p(x, t-\tau|y) \quad \text{for all} \quad t \geq \tau$$

in which case the probability distribution function is also stationary:

$$F(x, t|y, \tau) = P(x, t-\tau|y) \quad \text{for all} \quad t \geq \tau$$

For stationary Markov processes it is easy to show that the Chapman-Kolmogorov equations (3.5) and (3.7) become

$$p(x, t|y) = \int p(x, \tau|z)\, p(z, t-\tau|y)\, dz \quad \text{for all} \quad \tau \leq t \tag{3.8}$$

$$P(x, t|y) = \int P(x, \tau|z)\, p(z, t-\tau|y)\, dz \quad \text{for all} \quad \tau \leq t \tag{3.9}$$

Using the Chapman-Kolmogorov equation (3.5) and (3.7) we will derive partial differential equations for the evolution of $f(x, t|y, \tau)$ and $F(x, t|y, \tau)$.

## 3.2 Forward and Backward Equations

Take $\partial/\partial\tau$ of both sides of (3.5) to obtain *

$$\frac{\partial f(x, t|y, \tau)}{\partial\tau} = \int f(x, t|\xi, s)\, \frac{\partial f(\xi, s|y, \tau)}{\partial\tau}\, d\xi \tag{3.10}$$

Similarly taking $\partial/\partial\tau$ of (3.7) gives

$$\frac{\partial F(x, t|y, \tau)}{\partial\tau} = \int F(x, t|\xi, s)\, \frac{\partial f(\xi, s|y, \tau)}{\partial\tau}\, d\xi \tag{3.11}$$

Equations (3.10) and (3.11) are called backward equations and involve differentiation with respect to the backward time variable $\tau$.

The initial condition for (3.10) is

$$f(x, \tau|y, \tau) = \delta(x-y) \tag{3.12}$$

and the initial condition for (3.11) is

$$F(x, \tau|y, \tau) = 1(x, y) \tag{3.13}$$

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* Doob [12] shows that under certain continuity conditions on the process the required partial derivatives exist.

where the n-dimendional step function $1(y, x)$ is defined by

$$1(y, x) = \begin{cases} 1, & x_i \le y_i \quad \text{for all } i = 1, 2, \ldots, n \\ 0, & x_i > y_i \quad \text{for any } i = 1, 2, \ldots, n \end{cases} \tag{3.14}$$

By differentiating (3.5) with respect to the forward time variable $t$, a forward equation for the transition density can be obtained. A forward equation for transition distributions, however, can be obtained only under more restrictive conditions.

Useful differential forms of the forward and backward equation will now be derived for a scalar stationary Markov process. Following a standard procedure [13] consider the integral

$$\int \rho(x) \frac{\partial p(x, t|y)}{\partial t} dx$$

where $\rho(x)$ is an arbitrary analytic function of $x$. Using the definition of the derivative,

$$\int \rho(x) \frac{\partial p(x, t|y)}{\partial t} dx = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int \rho(x) \left[p(x, t + \Delta t|y) - p(x, t|y)\right] dx \tag{3.15}$$

Since, by (3.8)

$$p(x, t + \Delta t|y) = \int p(x, \Delta t|z)\, p(z, t|y)\, dz$$

and

$$\rho(x) = \sum_{i=0}^{\infty} \frac{\rho^{(i)}(z)}{i!} (x - z)^i$$

we have

$$\int \rho(x)\, p(x, t + \Delta t|y)\, dx = \int p(z, t|y) \sum_{i=0}^{\infty} \frac{\rho^{(i)}(z)}{i!} \int (x - z)^i\, p(x, \Delta t|z)\, dx\, dz$$

Define the $i$th conditional moment by

$$\mu_i(\Delta t|z) = \int (x - z)^i\, p(x, \Delta t|z)\, dx \tag{3.16}$$

$$\mu_0(\Delta t|z) = \int p(x, \Delta t|z)\, dx = 1$$

Then (3.15) becomes

$$\int \rho(x) \frac{\partial p(x, t|y)}{\partial t} dx = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int p(z, t|y) \sum_{i=1}^{\infty} \frac{\rho^{(i)}(z)}{i!} \mu_i(\Delta t|z)\, dz$$

$$= \sum_{i=1}^{\infty} \frac{1}{i!} \int \rho^{(i)}(z)\; \eta_i(z)\, p(z, t|y)\, dz \tag{3.17}$$

where

$$\eta_i(z) = \lim_{\Delta t \to 0} \frac{\mu_i(\Delta t|z)}{\Delta t} \tag{3.18}$$

provided this limit exists uniformly in $z$ for all $i$. Integration by parts of the integrals in (3.17) gives

$$\int \rho^{(i)}(x) \eta_i(x)\, p(x, t|y)\, dx = (-1)^i \int \rho(x) \frac{d^i}{dx^i} [\eta_i(x)\, p(x, t|y)]\, dx \tag{3.19}$$

since $p(x, t|z)$ is a density function and approaches zero with all its derivatives as $x \to \pm\infty$. Substituting (3.19) into (3.17) and noting that $\rho(x)$ is arbitrary we find that the transition density function satisfies

$$\frac{\partial p(x, t|y)}{\partial t} = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \frac{\partial^i}{\partial x^i} [\eta_i(x)\, p(x, t|y)] = \mathcal{L}_x^* [p(x, t|y)] \tag{3.20}$$

which is called the forward equation for $p(x, t|y)$ since it involves differentiation with respect to the forward time $t$ and a linear operation on the forward state $x$. Equation (3.20) is also called the "Boltzmann Equation" [14]. The differential operator $\mathcal{L}_x^*$ is called the forward operator of the process. The initial and boundary conditions for (3.20) are

$$p(x, 0|y) = \delta(x - y) \tag{3.21}$$

$$p(x, t|y) \to 0 \quad \text{as} \quad x \to \pm\infty \quad \text{for all } t.$$

A conditional expectation on a Markov process also satisfies a partial differential equation: consider the expectation

$$\bar{u}(t|y) = E\{u(X(t)) | X(0) = y\} = \int u(\lambda)\, p(\lambda, t|y)\, d\lambda \tag{3.22}$$

Then from the forward equation (3.20) we have

$$\int \bar{u}(\tau - s|y)\left[\frac{\partial p(y, s|x)}{\partial s} - \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \frac{\partial^i [\eta_i(y)\, p(y, s|x)]}{\partial y^i}\right] dy = 0 \tag{3.23}$$

Apply the definition (3.22) of $\bar{u}(\tau - s|y)$ to obtain

$$\int \bar{u}(\tau - s|y) \frac{\partial p(y, s|x)}{\partial s} dy = \int\int u(\lambda)\, p(\lambda, \tau - s|y) \frac{\partial p(y, s|x)}{\partial s} d\lambda\, dy \tag{3.24}$$

Now, from the Chapman-Kolmogorov equation (3.8)

$$\frac{\partial}{\partial s}\left[\int u(\lambda)\, p(\lambda, \tau - s|y)\, p(y, s|x)\, dy\right] = 0$$

Therefore,

$$\int u(\lambda) \frac{\partial p(\lambda, \tau - s|y)}{\partial s} p(y, s|x)\, dy = -\int u(\lambda)\, p(\lambda, \tau - s|y) \frac{\partial p(y, s|x)}{\partial s} dy$$

and (3.24) becomes

$$\int \bar{u}(\tau - s|y) \frac{\partial p(y, s|x)}{\partial s} dy = -\int \left[\frac{\partial}{\partial s}\int u(\lambda)\, p(\lambda, \tau - s|y)\, d\lambda\right] p(y, s|x)\, dy = \int \frac{\partial \bar{u}(\tau - s|y)}{\partial s} p(y, s|x)\, dy \tag{3.25}$$

Furthermore, from (3.19) we have

$$\int \bar{u}(\tau - s|y) \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \frac{\partial^i \eta_i(y)\, p(y, s|x)}{\partial y^i} dy = \int p(y, s|x) \sum_{i=1}^{\infty} \frac{\eta_i(y)}{i!} \frac{\partial^i u(\tau - s|y)}{\partial y^i} dy \tag{3.26}$$

Substitute (3.25) and (3.26) into (3.23) to give

$$\int p(y, s|x)\left[-\frac{\partial \bar{u}(\tau - s|y)}{\partial s} - \sum_{i=1}^{\infty} \frac{\eta_i(y)}{i!} \frac{\partial^i \bar{u}(\tau - s|y)}{\partial y^i}\right] dy = 0$$

Since $p(y, s|x) \geq 0$ and is not identically zero for all $y$,

$$-\frac{\partial \bar{u}(\tau - s|y)}{\partial s} = \sum_{i=1}^{\infty} \frac{\eta_i(y)}{i!} \frac{\partial^i \bar{u}(\tau - s|y)}{\partial y^i} \tag{3.27}$$

Now, let $\tau - s = t$ so that (3.27) becomes

$$\frac{\partial \bar{u}(t|y)}{\partial t} = \sum_{i=1}^{\infty} \frac{\eta_i(y)}{i!} \frac{\partial^i \bar{u}(t|y)}{\partial y^i} = \mathcal{L}_y[\bar{u}(t|y)] \tag{3.28}$$

which is the backward equation for $\bar{u}(t|y)$ since it involves a linear operation on the initial space variable $y$. The differential operator $\mathcal{L}_y$ is called the <u>backward operator</u> or <u>differential generator</u>

of the process. Note that by the definition of the adjoint [15] to a differential operator, $\mathcal{L}_x$ and $\mathcal{L}_x^*$ are adjoint operators. The initial condition for (3.28) is $\bar{u}(0|y) = \bar{u}(y)$ a known function of $y$. It is important to note that the transition distribution function $P(x, t|y)$ is a particular expectation, i.e.,

$$P(x, t|y) = \int 1(x, \lambda)\, p(\lambda, t|y)\, d\tau$$

where $1(x, \lambda)$ is defined in (3.14). Similarly, the transition density satisfies the backward equation.

An important class of Markov processes is that in which the limits $\eta_l(z)$ vanish for $l > 2$. The forward equation (3.20) for such processes becomes

$$\frac{\partial p(x, t|y)}{\partial t} = -\frac{\partial}{\partial x}[\eta_1(x)\, p(x, t|y)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\eta_2(x)\, p(x, t|y)] \tag{3.29}$$

which is known as the Fokker-Planck equation. The corresponding backward equation is

$$\frac{\partial \bar{u}(t|y)}{\partial t} = \eta_1(y)\frac{\partial \bar{u}(t|y)}{\partial y} + \frac{1}{2}\eta_2(y)\frac{\partial^2 \bar{u}(t|y)}{\partial y^2} \tag{3.30}$$

For the Wiener process having the transition density (2.10), it is found that $\eta_2(x) = \sigma^2$ and $\eta_l(x) = 0$ for $l \neq 2$. Hence the forward and backward equations both reduce to the diffusion equation

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} \tag{3.31}$$

which was obtained by Einstein in his study of Brownian motion.

For vector processes, (3.29) and (3.30) can be generalized to

$$\frac{\partial p(x, t|y)}{\partial t} = -\sum_l \frac{\partial}{\partial x_l}(h_l(x)\, p(x, t|y)) + \frac{1}{2}\sum_{l,j}\frac{\partial^2}{\partial x_l\,\partial x_j}(H_{lj}(x)\, p(x, t|y)) \tag{3.32}$$

and

$$\frac{\partial \bar{u}(t|y)}{\partial t} = \sum_l h_l(y)\frac{\partial \bar{u}(t|y)}{\partial y_l} + \frac{1}{2}\sum_{l,j} H_{lj}(y)\frac{\partial^2 \bar{u}(t|y)}{\partial y_l\,\partial y_j} \tag{3.33}$$

where the vector $h$ and the matrix $H$ are defined analogously to (3.18)

$$h(z) = \lim_{\Delta t \to 0}\frac{1}{\Delta t}\int (x - z)\, p(x, \Delta t|z)\, dx \tag{3.34}$$

$$H(z) = \lim_{\Delta t \to 0}\frac{1}{\Delta t}\int (x - z)(x - z)'\, p(x, \Delta t|z)\, dx \tag{3.35}$$

provided the limits analogous to $\eta_i(z)$ (which are tensors of rank $i$ in vector processes) vanish for $i > 2$.

For the Poisson step process with density (2.4) the forward equation can be found from (3.20) and (3.18). The moments $\mu_i(\Delta t|z)$ of $p(x, \Delta t|z)$ about $z$ are obtained by inserting (2.4) into (3.16) to give

$$\mu_i(\Delta t|z) = \sum_{n=0}^{\infty} \frac{(\lambda\Delta t)^n}{n!} e^{-\lambda\Delta t} \int (x - z)^i \gamma_n(x - z)\,dx \tag{3.36}$$

By denoting the integral in (3.36) as $\mu_i$ (which is independent of $z$) we get from the definition (3.18)

$$\eta_i = \lambda\mu_i$$

Hence (3.20) gives

$$\frac{\partial p(x, t|y)}{\partial t} = \lambda \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \mu_i \frac{\partial^i p(x, t|y)}{\partial x^i} \tag{3.37}$$

as the forward equation. The corresponding backward equation from (3.28) is thus

$$\frac{\partial \bar{u}(t|y)}{\partial t} = \lambda \sum_{i=1}^{\infty} \frac{\mu_i}{i!} \frac{\partial^i \bar{u}(t|y)}{\partial y^i} \tag{3.38}$$

These same results can be obtained in an alternate manner from (3.9) as follows. Substitute (2.4) into (3.9) to obtain

$$P(x, t|y) = \int P(x, \tau|z)\, e^{-\lambda(t-\tau)}\, \delta(z - y)\,dz + \int P(x, \tau|z)\, e^{-\lambda(t-\tau)} \sum_{n=1}^{\infty} \frac{[\lambda(t-\tau)]^n}{n!} \gamma_n(z - y)\,dz \tag{3.39}$$

Since $P(x, t|y)$ is independent of the intermediate time $\tau$, we have $\partial P(x, t|y)/\partial\tau = 0$ and from (3.39)

$$\begin{aligned} 0 = \lambda P(x, \tau|y) + \frac{\partial P(x, \tau|y)}{\partial\tau} &+ \int \left[\frac{\partial P(x, \tau|z)}{\partial\tau} + P(x, \tau|z)\lambda\right] \sum_{n=1} \frac{[\lambda(t-\tau)]^n}{n!} \gamma_n(z - y)\,dz \\ &+ \int P(x, \tau|z) \sum_{n=1}^{\infty} \frac{n[\lambda(t-\tau)]^{n-1}(-\lambda)}{n!} \gamma_n(z - y)\,dz \end{aligned} \tag{3.40}$$

Upon setting $\tau = t$ we obtain the result

$$\frac{\partial P(x, t|y)}{\partial t} = -\lambda P(x, t|y) + \lambda \int P(x, t|z)\, \gamma_1(z - y)\,dz \tag{3.41}$$

which is the integral form of the backward equation. Since the transition density $p(x, t|y)$ also satisfies the backward equation

$$\frac{\partial p(x, t|y)}{\partial t} = -\lambda p(x, t|y) + \lambda \int p(x, t|z)\, \gamma_1(z - y)\, dz \tag{3.42}$$

multiply both sides of (1) by $p(y, \tau - t|x)$ and integrate with respect to y to obtain

$$\begin{aligned}\int \frac{\partial p(x, t|y)}{\partial t} p(y, \tau - t|x)\, dy &= -\lambda \int p(x, t|y)\, p(y, \tau - t|x)\, dy \\ &\quad + \lambda \int p(x, t|z)\left[\int \gamma_1(z - y)\, p(y, \tau - t|x)\, dy\right] dz\end{aligned} \tag{3.43}$$

Since

$$\frac{\partial}{\partial t}\int p(x, t|y)\, p(y, \tau - t|x)\, dy = \frac{\partial}{\partial t} p(x, \tau|x) = 0$$

we have

$$\int \frac{\partial p(x, t|y)}{\partial t} p(y, \tau - t|x)\, dy = -\int p(x, t|y)\, \frac{\partial p(y, \tau - t|x)}{\partial t}\, dy$$

and (3.43) becomes

$$\begin{aligned}-\int p(x, t|y)\, \frac{\partial p(y, \tau - t|x)}{\partial t}\, dy &= -\lambda \int p(x, t|y)\, p(y, \tau - t|x)\, dy \\ &\quad + \lambda \int p(x, t|y)\left[\int \gamma_1(y - z)\, p(z, \tau - t|x)\, dz\right] dy\end{aligned} \tag{3.44}$$

Since $p(x, t|y)$ is not identically zero (3.44) implies

$$-\frac{\partial p(y, \tau - t|x)}{\partial t} = -\lambda p(y, \tau - t|x) + \lambda \int p(z, \tau - t|x)\, \gamma_1(y - z)\, dz$$

which, upon relabeling variables, may be written as the forward equation

$$\frac{\partial p(x, t|y)}{\partial t} = -\lambda p(x, t|y) + \lambda \int p(\eta, t|y)\, \gamma_1(x - \eta)\, d\eta \tag{3.45}$$

A differential form of the forward equation (3.45) may be obtained as follows. Write (3.45) as

$$\frac{\partial p(x, t|y)}{\partial t} = -\lambda p(x, t|y) + \lambda \int p(x - \eta, t|y)\, \gamma_1(\eta)\, d\eta \tag{3.46}$$

Expand $p(x - \eta, t|y)$ in Taylor series about $\eta = 0$ and insert into (3.46) to give the forward equation

$$
\begin{aligned}
\frac{\partial p(x, t|y)}{\partial t} &= -\lambda\, p(x, t|y) + \lambda \int \sum_{i=0}^{\infty} (-1)^{i} \frac{\partial^{i} p(x, t|y)}{\partial x^{i}} \frac{\eta^{i}}{i!} \gamma_1(\eta)\, d\eta \\
&= \lambda \sum_{i=1}^{\infty} (-1)^{i} \frac{m_i}{i!} \frac{\partial^{i} p(x, t|y)}{\partial x^{i}}
\end{aligned}
\tag{3.47}
$$

where $m_i$ is the $i^{\underline{th}}$ moment of the density $\gamma_1(\eta)$. Note that in a similar manner we can show that the backward equation (3.42) leads to the differential form

$$
\frac{\partial p(x, t|y)}{\partial t} = \lambda \sum_{i=1}^{\infty} \frac{m_i}{i!} \frac{\partial^{i} p(x, t|y)}{\partial y^{i}} \tag{3.48}
$$

### 3.3 Confinement Probability; Renewal Equation

A function which is useful in the study of stability of randomly excited systems is called the "confinement probability" defined as

$$
q(t|y) = \text{Prob } [X(s) \in N \quad \text{for all} \quad s \in [0, t] \,|\, X(0) = y] \tag{3.49}
$$

where $N$ is a finite region containing the origin. We will show that for a scalar Markov process $q(t|y)$ satisfies the backward equation (3.11).

Our demonstration will use the renewal equation for scalar Markov processes. Consider a first-order, continuous Markov process $\{X(t)\}$ with transition density function $f(x, t|y, v)$ and transition distribution function $F(x, t|y, v)$. Let the random variable $T(z|y, v)$ be the first time that the state of the process is $z$ given that the state was $y$ at time $v$. Let $R(z, \tau|y, v)$ and $r(z, \tau|y, v)$ be the distribution function and density function, respectively, of $T(z|y, v)$, i.e.,

$$
R(z, \tau|y, v) = \text{Prob } \{T(z|y, v) < \tau\} \tag{3.50}
$$

$$
r(z, \tau|y, v)\, d\tau = \text{Prob } \{\tau < T(z|y, v) < \tau + d\tau\} \tag{3.51}
$$

Divide the time interval $[t - v]$ into $n$ sub-intervals $\tau_i < t \le \tau_{i+1}$, $i = 0, 1, \ldots, n-1$, where $\tau_0 = v$ and $\tau_n = t$. Let $t_1, t_2, \ldots, t_n$ correspond to points inside these sub-intervals. Then for any $z$ $(x \lesseqgtr z \gtreqless y)$

$$\text{Prob}\{x < X(t) < x + dx \mid X(v) = y\}$$

$$= \text{Prob}\{x < X(t) < x + dx \mid X(t_1) = z\} \cdot \text{Prob}\{\tau_0 < T(z \mid y, v) \le \tau_1\} \tag{3.52}$$

$$+ \text{Prob}\{x < X(t) < x + dx \mid X(t_2) = z\} \cdot \text{Prob}\{\tau_1 < T(z \mid y, v) \le \tau_2\}$$

$$+ \ldots + \text{Prob}\{x < X(t) < x + dx \mid X(t_n) = z\} \cdot \text{Prob}\{\tau_{n-1} < T(z \mid y, v) \le \tau_n\}$$

Equation (3.52) is based on the assumption that $\{X(t)\}$ is a continuous Markov process and that the occurrence of the state $z$ for the first time in one sub-interval is mutually exclusive of the occurrence of $z$ for the first time in another sub-interval. Rewrite (3.52) as

$$\text{Prob}\{x < X(t) < x + dx \mid X(v) = y\} = \sum_{t_l = v}^{t_n = t} \text{Prob}\{x < X(t) < x + dx \mid X(t_l) = z\} \cdot$$

$$\cdot \text{Prob}\{\tau_{l-1} < T(z \mid y, v) \le \tau_l\} \tag{3.53}$$

$$= \sum_{t_l = v}^{t_n = t} \text{Prob}\{x < X(t) < x + dx \mid X(t_l) = z\} \text{Prob}\{\tau_{l-1} < T(z \mid y, v) \le \tau_{l-1} + \Delta_l\}$$

where $\Delta_l = \tau_l - \tau_{l-1}$. Therefore, in the limit as $n \to \infty$ and $\Delta_l \to 0$, (3.53) reduces to the renewal equation [16]

$$f(x, t \mid y, v) = \int_v^t f(x, t \mid z, \tau)\, r(z, \tau \mid y, v)\, d\tau \tag{3.54}$$

Notice that (3.54) may be considered to be a convolution in time whereas the Chapman-Kolmogorov equations (3.5) and (3.7) may be regarded as convolutions in space.

We will use (3.54) to show that if $F(x, t \mid y, v)$ satisfies the backward equation (3.11) then $r(z, \tau \mid y, v)$ and $R(z, \tau \mid y, v)$ satisfy the backward equation

$$\frac{\partial r(z, \tau \mid y, v)}{\partial v} = \int r(z, \tau \mid \xi, s) \frac{\partial f(\xi, s \mid y, v)}{\partial v}\, d\xi \tag{3.55}$$

Multiply both sides of (3.54) by a unit step (3.14) and integrate over the state space to obtain

$$F(x, t \mid y, v) = \int_v^t F(x, t \mid z, \tau)\, r(z, \tau \mid y, v)\, d\tau \tag{3.56}$$

Substitute (3.56) into both sides of (3.11). The left hand side of (3.11) becomes

$$\int_v^t F(x, t|z, \tau) \frac{\partial r(z, \tau|y, v)}{\partial v} d\tau - F(x, t|z, v)\, r(z, v|y, v) \tag{3.57}$$

Note that $r(z, v|y, v) = 0$ for $z \neq y$, so that (3.57) becomes

$$\int_v^t F(x, t|z, \tau) \frac{\partial r(z, \tau|y, v)}{\partial v} d\tau \tag{3.58}$$

The right hand side of (3.11) is

$$\int\int_v^t [F(x, t|z, \tau)\, r(z, \tau|\xi, s)\, d\tau] \frac{\partial f(\xi, s|y, v)}{\partial v} d\xi \tag{3.59}$$

Interchange the order of integration in (3.59) to obtain

$$\int_v^t F(x, t|z, \tau)[\int r(z, \tau|\xi, s) \frac{\partial f(\xi, s|y, v)}{\partial v} d\xi]\, d\tau \tag{3.60}$$

Now, equate (3.58) and (3.60), and since $v$ and $t$ are arbitrary, obtain

$$\frac{\partial r(z, \tau|y, v)}{\partial v} = \int r(z, \tau|\xi, s) \frac{\partial f(\xi, s|y, v)}{\partial v} d\xi \tag{3.61}$$

which is (3.55). Further by the definition of distribution and density functions,

$$R(z, \tau|y, v) = \int_v^\tau r(z, \eta|y, v)\, d\eta \tag{3.62}$$

Therefore, integrate both sides of (3.61) with respect to the forward time-variable $\tau$ to obtain the backward equation for $R(z, \tau|y, v)$.

$$\frac{\partial R(z, \tau|y, v)}{\partial v} = \int R(z, \tau|\xi, s) \frac{\partial f(\xi, s|y, v)}{\partial v} d\xi \tag{3.63}$$

Now, for scalar systems, consider the problem of determining

$$q(t|y, t_0) = \text{Prob}\{ \sup_{t_0 \le \tau \le t} |X(\tau)| < B | X(t_0) = y\}, \quad -B < y < B \tag{3.64}$$

where $B$ is some positive number. Then, for $|y| < B$,

$$1 - q(t|y, t_0) = \text{Prob}\,\{\sup_{t_0 \le \tau \le t} |X(\tau)| \ge B\} \tag{3.65}$$

Define the events $\mathcal{E}_1$ and $\mathcal{E}_2$,

$\mathcal{E}_1$ : the event that $X(\tau) = B$

$\mathcal{E}_2$ : the event that $X(\tau) = -B$

and the random variables $T_1$, $T_2$, and $\hat{T}$,

$T_1(\mathcal{E}_1|y, t_0)$ : the first time that $\mathcal{E}_1$ occurs, given $X(t_0) = y$

$T_2(\mathcal{E}_2|y, t_0)$ : the first time that $\mathcal{E}_2$ occurs, given $X(t_0) = y$

$\hat{T}(\mathcal{E}_1, \mathcal{E}_2|y, t_0)$ : the first time that both $\mathcal{E}_1$ and $\mathcal{E}_2$ occur, given $X(t_0) = y$

Then

$$\text{Prob}\,\{\sup_{t_0 \le \tau \le t} |X(\tau)| \ge B\} = \text{Prob}\,\{T_1 < t\} + \text{Prob}\,\{T_2 < t\} - \text{Prob}\,\{\hat{T} < t\} \tag{3.66}$$

and (3.66) becomes

$$q(t|y, t_0) = 1 - R(B, t|y, t_0) - R(-B, t|y, t_0) + \hat{R}(B, -B, t|y, t_0) \tag{3.67}$$

where

$$\hat{r}(B, -B, t|y, t_0)\,dt = \text{Prob}\,\{t < \hat{T} < t + dt\} \tag{3.68}$$

and

$$\hat{R}(B, -B, t|y, t_0) = \int_{t_0}^{t} \hat{r}(B, -B, \tau|y, t_0)\,d\tau \tag{3.69}$$

Now, the first three terms on the right-hand side of (3.67) satisfy the backward equation. We will show that $\hat{R}(B, -B, t|y, t_0)$ also satisfies the backward equation; consequently, $q(t|y, t_0)$ is a linear combination of solutions and thus satisfies the same backward equation. Divide the time interval $[t_0, t]$ into n sub-intervals $\tau_\iota < t \le \tau_{\iota+1}$, $\iota = 0, 1, \ldots, n-1$, where $\tau_0 = t_0$ and $\tau_n = t$. Let $t_1, t_2, \ldots, t_n$ correspond to points inside these sub-intervals. Then

$$\text{Prob}\{t < \hat{T} < t + dt\} = \text{Prob}(t < T_2 < t + dt \mid \mathcal{E}_1 \text{ occurs at } t_1) \cdot \text{Prob}(\tau_0 < T_1 < \tau_1 | y, t_0) + \ldots$$

$$+ \text{Prob}(t < T_2 < t + dt | \mathcal{E}_1 \text{ occurs at } t_n) \cdot \text{Prob}(\tau_{n-1} < T_1 < \tau_n | y, t_0)$$

$$+ \text{Prob}(t < T_1 < t + dt | \mathcal{E}_2 \text{ occurs at } t_1) \cdot \text{Prob}(\tau_0 < T_2 < \tau_1 | y, t_0)$$

$$+ \ldots \text{Prob}(t < T_1 < t + dt | \mathcal{E}_2 \text{ occurs at } t_2) \cdot \text{Prob}(\tau_{n-1} < T_2 < \tau_n | y, t_0)$$

Now, writing $\Delta_\ell = \tau_\ell - \tau_{\ell-1}$ and taking the limit as $n \to \infty$ and $\Delta_\ell \to 0$, we have

$$\hat{r}(B, -B, t|y, t_0) = \int_{t_0}^{t} r(-B, t|B, \tau)\, r(B, \tau|y, t_0)\, d\tau + \int_{t_0}^{t} r(B, t|-B, \tau)\, r(-B, \tau|y, t_0)\, d\tau \tag{3.70}$$

Now, we obtain directly from (3.70),

$$\frac{\partial \hat{r}(B, -B, t|y, t_0)}{\partial t_0} = \int_{t_0}^{t} r(-B, t|B, \tau) \frac{\partial r(B, \tau|y, t_0)}{\partial t_0} d\tau - r(-B, t|B, t_0)\, r(B, t_0|y, t_0) + \int_{t_0}^{t} r(B, t|-B, \tau) \frac{\partial r(-B, \tau|y, t_0)}{\partial t_0} d\tau - r(B, t|-B, t_0)\, r(-B, t_0|y, t_0) \tag{3.71}$$

Note that $r(B, t_0|y, t_0) = 0$ for $y \neq B$ and $r(-B, t_0|y, t_0) = 0$ for $y \neq -B$, so that (3.71) becomes

$$\frac{\partial \hat{r}(B, -B, t|y, t_0)}{\partial t_0} = \int_{t_0}^{t} r(-B, t|B, \tau) \frac{\partial r(B, \tau|y, t_0)}{\partial t_0} d\tau + \int_{t_0}^{t} r(B, t|-B, \tau) \frac{\partial r(-B, \tau|y, t_0)}{\partial t_0} d\tau \tag{3.72}$$

Now, since $r(B, \tau|y, t_0)$ and $r(-B, \tau|y, t_0)$ satisfy the backward equation, (3.72) may be written

$$\frac{\partial \hat{r}(B, -B, t|y, t_0)}{\partial t_0} = \int_{t_0}^{t} r(-B, t|B, \tau) \left( \int r(B, \tau|\xi, s) \frac{\partial p(\xi, s|y, v)}{\partial v} d\xi \right) d\tau + \int_{t_0}^{t} r(B, t|-B, \tau) \left( \int r(-B, \tau|\xi, s) \frac{\partial p(\xi, s|y, v)}{\partial v} d\xi \right) d\tau \tag{3.73}$$

Interchanging the order of integration and using (3.69), we obtain finally

$$\frac{\partial \hat{R}(B, -B, t|y, t_0)}{\partial t_0} = \int \hat{R}(B, -B, t|\xi, s) \frac{\partial p(\xi, s|y, v)}{\partial v} d\xi \tag{3.74}$$

Thus $\hat{R}(B, -B, t|y, t_0)$ satisfies the backward equation and $q(t|y, t_0)$ does also.

A general proof that $q(t|y)$ of (3.49) satisfies the backward equation for vector random processes is given in Dynkin [4]. (Since $q(t|y)$ can be interpreted as the expectation of a functional such as $\sup_{0 \le s \le t} \|X(s)\|$ on the output Markov process $\{X(t)\}$, the fact that $q$ satisfies the backward equation is plausible if the result for expectations of functions of Markov processes holds also for functionals.)

# 4. RANDOMLY-EXCITED DYNAMIC SYSTEMS

## 4.1 Stochastic Differential Equations

The first problem that arises in the study of randomly-excited dynamic systems is characterization. Consider a system which, when subjected to a deterministic input $\xi(t)$, can be represented by the system of ordinary differential equations

$$\dot{x} = f(x, \xi, t) \tag{4.1}$$

where $x$ is the state of the system. When the excitation $\xi(t)$ is not deterministic, but is a sample function of a stochastic process $\{\xi(t)\}$ then the state $x(t)$ for $t \geq t_0$ which evolves according to (4.1) from a known initial state $x_0$ will depend on the particular sample function applied. Therefore $x$ can be considered as a sample function of a stochastic process $\{x(t)\}$. A suitable description of a randomly excited dynamic system syould enable one to determine the characteristics of the output stochastic process $\{x(t)\}$ given the characteristics of the input process $\{\xi(t)\}$.

Certain difficulties of rigor can arise, however, when the very general system (4.1) is considered. This is due to the fact that for some random excitation processes $\{\xi(t)\}$ such as Gaussian white noise (a process characterized by the properties $E\{\xi\} = 0$ and $E\{\xi(t)\xi'(\tau)\} = \delta(t-\tau)\Sigma$ where $\Sigma$ is a non-singular matrix) the derivative in (4.1) is not well-defined.

Because of these difficulties, most investigations have been concerned with the "additive" system

$$\dot{x} = g(x, t) + G(x, t)\xi \tag{4.2}$$

where $\{\xi\}$ is a white noise process. (It is shown below that one can interpret a Wiener process as being generated by integrating the sample functions of a stationary white noise process.) To make (4.2) meaningful, Itô [17] replaced (4.2) by the stochastic differential equation

$$dx = g(x, t)\,dt + G(x, t)\,dw \tag{4.3}$$

where dw is a sample function of a Wiener process. Eq. (4.3) in turn is defined to be equivalent to the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} g(x(\tau), \tau)\, d\tau + \int_{t_0}^{t} G(x(\tau), \tau)\, dw\,(\tau) \tag{4.4}$$

where the second, "stochastic", integral is defined in a special way, namely

$$\int_{t_0}^{t} G(x(\tau), \tau)\, dw\,(\tau) = \text{l.i.m.} \sum_k G(x(t_k), t_k)[w(t_{k+1}) - w(t_k)] \tag{4.5}$$

as $\max_k [t_{k+1} - t_k] \to 0$. In this way, a rigorous, self-consistent theory of stochastic, differential equations has been established. However, Wong and Zakai [18] have shown that if the dw process is replaced by a sequence of well-behaved processes $y_n$ so that (4.3) has the solution $x_n$ in the sense of ordinary differential equations and if $y_n$ converges to w as $n \to \infty$ then $x_n$ does not converge to the solution of (4.4) when Itô's definition (4.5) of the second (stochastic) integral is used. Since this limiting approach corresponds to the actual experimental treatment of physical systems, this inconsistency between the physical and rigorous mathematical treatment is unfortunate.

Gersch [19] has also pointed out that this same problem exists for deterministic systems with delta-function parameter variations. Of course, delta-functions and white noise are mathematical idealizations and cannot occur in actual physical systems.

To avoid the inconsistency, Stratonovich has given an alternate definition of the stochastic integral in (4.4) which can also be used to establish a rigorous mathematical theory and which seems to be consistent with physical experiments. (See Wonham [20].) According to Stratonovich, the stochastic integral in (4.4) is to be interpreted as

$$\int_{t_0}^{t} G(x(\tau), \tau)\, dw\,(\tau) = \text{l.i.m.} \left[\sum_k G\left(\frac{x(t_{k+1}) + x(t_k)}{2}, t_k\right)\right] [w(t_{k+1}) - w(t_k)] \tag{4.6}$$

as $\max_k [t_{k+1} - t_k] \to 0$. We note that when G does not depend on x there is no difference between Itô's and Stratonovich's definitions so no difficulty arises in using the form (4.2) to describe the system.

For simplicity in the sequel, we consider only the system

$$\dot{x} = g(x) + G\xi \tag{4.7}$$

where $G$ is a constant matrix. Equation (4.7) has the form of the Langevin equation [3] which has been extensively studied by physicists in connection with Brownian motion.

## 4.2 Integrator Excited by White Noise

Consider the simplest case of (4.7): $x$ is a scalar, $g = 0$, $G = 1$, and $\xi$ is white noise, which has the properties $E(\xi(t)) = 0$, $E(\xi(t)\,\xi(\tau)) = \sigma^2\,\delta(t - \tau)$. Formal integration of (4.7) results in

$$x(t) = y + \int_0^t \xi(\tau)\, d\tau \qquad\qquad x(0) = y$$

Hence

$$E(x(t)\,|\,y) = y$$

and

$$E(x^2(t)\,|\,y) = y^2 + \int_0^t \int_0^t \sigma^2 \delta(\tau - \eta)\, d\tau\, d\eta = y^2 + \sigma^2 t$$

The variance of $x^2(t)$ given $y$ thus is $\sigma^2 t$. The Gaussian variable having a mean $y$ and a variance $\sigma^2 t$ characterizes the Wiener process described earlier. Hence a Wiener process may be regarded as the output of an integrator, starting in the state $y$ and excited by white noise with a correlation function $\sigma^2\delta(t - \tau)$. More generally, if a vector white noise process $\xi$ ($E(\xi) = 0$, $E(\xi(t)\,\xi'(\tau)) = \Sigma\,\delta(t - \tau)$ where $\Sigma$ is the "covariance matrix") is integrated, so that $\dot{x} = \xi$ with $x_0 = y$, it is found that $E(x(t)) = y$, $\mathrm{Var}\,[x(t)\,x'(t)\,|\,y] = \Sigma t$ and

$$p(x, t\,|\,y) = ((2\pi)^n \det \Sigma t)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}(x - y)'(\Sigma t)^{-1}(x - y)\right) \tag{4.8}$$

where $n$ is the dimension of the vectors $x$ and $y$. Hence the output process $x$ can be regarded as a multidimensional Wiener process.

## 4.3 Additive System Excited by White Noise

Now suppose that white noise is the input $\xi$ to the system (4.7) which starts in the state $x_0 = y$. We can write

$$dx = g(x)\, dt + G\, dw \ , \quad x_0 = z \tag{4.9}$$

where $dw/dt = \xi$ , i.e., $dw$ is a Wiener process having the density function (4.8) with $y = 0$ . Then

$$E\{dx \mid x_0 = z\} = g(z)\, dt + GE\{dw\}$$

and since $E\{dw\} = 0$ for a Wiener process, we obtain the vector $h(z)$ of (3.34) as

$$h(z) = g(z) \tag{4.10}$$

Similarly

$$E\{(dx)(dx)' \mid x_0 = z\} = g(z)\, g'(z)(dt)^2 + GE\{(dw)(dw)'\}\, G'$$

and hence the matrix $H(z)$ of (3.35) is

$$H(z) = G \Sigma G' \equiv D \equiv [D_{ij}] \tag{4.11}$$

since $E\{(dw)(dw)'\} = \Sigma\, dt$ for the Wiener process $dw$ .

It turns out that the analogs of the $\eta_i$ for $i > 2$ vanish in this case. However, since tensors are involved we shall demonstrate this for a scalar process. We have

$$\begin{aligned} E\{(dx)^k \mid x_0 = z\} &= E\{(g(z)\, dt + dw)^k\} \\ &= E\{(dw)^k\} + kg(z)\, E\{(dw)^{k-1}\}\, dt + o\,(dt) \end{aligned}$$

Now, the Wiener process, with the density function (4.8), $E\{(dw)^r\} = 1 \cdot 3 \cdot \ldots \cdot (k-1)(dt)^{r/2}$ for $r$ even and is zero for $r$ odd; hence, for $k \leq 2$, $E\{(dw)^k\} = o\,(dt)$ and $E\{(dw)^{k-1}\}\, dt = o\,(dt)$. Thus $\eta_k(z) = 0$ for $k > 2$ . A similar calculation reveals that the multidimensional analogs of $\eta_k(z)$ vanish for $k > 2$ .

Upon substitution of (4.10) and (4.11) into the general form of the forward equation (3.32) we obtain

$$\frac{\partial p(x\, ,\, t|y)}{\partial t} = \sum_i \frac{\partial}{\partial x_i}\, [g_i(x)\, p(x\, ,\, t|y)] + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i\, \partial x_j}\, [D_{ij}\, p(x\, ,\, t|y)] \tag{4.12}$$

If $g(x) = Ax$ , then the solution to (4.12) is found to be

$$p(x\, ,\, t|y) = \frac{1}{(2\pi)^{n/2}\, [\det M]^{\frac{1}{2}}} \exp[-\tfrac{1}{2}(x-\mu)' M^{-1} (x-\mu)] \tag{4.13}$$

where

$$\dot{\mu} = A\mu \ , \ \mu(0) = y \tag{4.14}$$

$$\dot{M} = AM + MA' + G\Sigma G' \qquad M(0) = 0 \tag{4.15}$$

Direct substitution of (4.13) into (4.12) verifies that this is the solution. It is thus noted that the transition probability density function of the state of a linear system perturbed by Gaussian white noise is also Gaussian with a mean $\mu(t)$ which satisfies the unexcited differential equation of the process, and with a covariance matrix governed by the variance equation (4.15). The backward equation corresponding to (4.12), upon use of 3.33), is given by

$$\frac{\partial \bar{u}(t\,|\,y)}{\partial t} = \sum_{i=1} g(y) \frac{\partial \bar{u}(t\,|y)}{\partial y_i} + \tfrac{1}{2} \sum_i \sum_j D_{i,j} \frac{\partial^2 \bar{u}\,(t|y)}{\partial y_i \, \partial y_j} \qquad D = G\Sigma G' \tag{4.16}$$

or, more simply

$$\frac{\partial \bar{u}\,(t\,|\,y)}{\partial t} = g(y) \cdot \nabla_y \bar{u}(t\,|y) + \tfrac{1}{2} \nabla_y \cdot [D \nabla_y \bar{u}(t\,|y)] \tag{4.17}$$

In particular, the conditional characteristic function is an expectation, i.e.,

$$C(\nu, t|y) = E\{e^{j\nu' x}|\,y\} = \int e^{j\nu' x} p(x\,,\,t|y)\,dx \tag{4.18}$$

and must be a solution to (4.17). The characteristic function corresponding to (4.13) is well-known as

$$C(\nu, t|\,y) = \exp\,[-\tfrac{1}{2}\,\nu' M \nu + j\nu'\mu] \tag{4.19}$$

and consequently (4.19) is a solution to (4.17). Direct substitution of (4.19) into (4.17) verifies the fact.

## 4.4 First-Order System Excited by a Markov Process

Consider the first order dynamical system

$$dx = g(x)\,dt + d\xi \tag{4.20}$$

where $d\xi$ is now a sample function of a Markov process with conditional moments $\mu_i(\Delta t|z)$ given by (3.16). When (4.20) is integrated over a small interval $\Delta t$ we have

$$\Delta x = g(x)\,\Delta t + \Delta\xi \tag{4.21}$$

Therefore, the moments of the output are given by

$$\hat{\mu}_i(\Delta t) = E(\Delta x)^i = E[g(x)\,\Delta t + \Delta\xi]^i \tag{4.22}$$

$$= \sum_{k=0}^{i} \binom{i}{k} [g(x)\,\Delta t]^k \mu_{i-k}(\Delta t|z), \qquad i \geq 1 \tag{4.23}$$

So that

$$\hat{\eta}_i(z) = \lim_{\Delta t \to 0} \frac{\hat{\mu}_i(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \sum_{k=0}^{i} \binom{i}{k} [g(x)\,\Delta t]^k \frac{\mu_{i-k}(\Delta t|z)}{\Delta t} \tag{4.24}$$

and

$$\begin{aligned} \hat{\eta}_1(z) &= g(x)\,\mu_0(0|z) + \eta_1(z) \\ &= g(x) + \eta_1(z) \end{aligned} \tag{4.25}$$

$$\hat{\eta}_i(z) = \eta_i(z) \qquad\qquad i \geq 2. \tag{4.26}$$

Therefore, the general expression of the forward equation (3.20) becomes

$$\frac{\partial p(x,t|y)}{\partial t} = -\frac{\partial[g(x)\,p(x,t|y)]}{\partial x} + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \frac{\partial^i \eta_i(x)\,p(x,t|y)}{\partial x^i} \tag{4.27}$$

Note that the first term on the right-hand side of (4.12) and (4.27) is a drift term which represents the effect of the system dynamics on the input process.

We have found in Section 3 that if $d\xi$ is a (4.7) Poisson step process, i.e., $d\xi/dt$ is a Poisson impulse process, $\eta_i(x) = \lambda m_i$ independent of x for $i \geq 2$, where the $m_i$ represents the moments of the amplitude distribution. Thus, (4.27) becomes

$$\frac{\partial p(x,t|y)}{\partial t} = -\frac{\partial[g(x)\,p(x,t|y)]}{\partial x} + \lambda \sum_{k=1}^{\infty} (-1)^k \frac{m_k}{(k)!} \frac{\partial^k p(x,t|y)}{\partial x^k} \tag{4.28}$$

Equation (4.28) is the forward equation for the probability density of the output of the first order system perturbed by Poisson impulse process.

Note that because of the presence of terms $\eta_i(x)$ for $i > 2$, the Poisson impulse process is more difficult to generalize to n-dimensions than is the Wiener process. This is one reason that the Wiener process has found widespread use in applications.

## 5. STABILITY AND LIFETIME

### 5.1 Definitions of Stability

It will be recalled that an equilibrium state of deterministic dynamic system is said to be stable if for any arbitrary neighborhood of the equilibrium state it is possible to find another neighborhood such that if the system starts out in the second neighborhood it always remains within the first. This concept of stability can be stated in probabilistic terms even though the system is deterministic. Suppose that $x(t)$ is the state of the system at time $t$ and that the origin is the equilibirum state. We can say that the origin is stnble if and only if for every neighborhood $N$ of the origin there exists a neighborhood $M$ of the origin such that

$$q(y) = \text{Prob } [X(s) \in N \text{ for all } s > 0 | X(0) = y] = 1$$

for any $y$ in $M$. A stability definition of this type, which may be termed stability with probability one or almost sure stability, can be applied directly to randomly-excited systems.

If we let $\|X\|$ denote the norm of the vector $X$, we have the following formal definitions of almost sure stability (asymptotic stability):

Definition 1 - AS. The origin is stable with probability one (w.p. 1) if and only if to any $\epsilon > 0$ there corresponds a $\delta(\epsilon)$ such that if $\|X(t_0)\| < \delta$, then

$$\text{Prob } (\sup_{t>t_0} \|X(t)\| > \epsilon) = 1 .$$

Definition 2 - AS. The origin is asymptotically stable with probability one if it is stable w.p. 1 and if for any $\mu > 0$ there exists a time $t_\mu$ such that

$$\text{Prob } (\sup_{t>t_\mu} \|X(t)\| < \mu) = 1 .$$

The definitions of stability with probability one are very strong, since they entail, in essence, the examination of every member of the output ensemble $\{X(t)\}$. Kushner [7] considered a slightly weaker definition of stochastic stability, namely

Definition 1 - AS'. The origin is stable with probability one (w.p. 1)* if and only if to any $0 < \rho < 1$, $\epsilon > 0$, there corresponds a $\delta(\rho, \epsilon)$ such that if $\|X(t_0)\| < \delta$, then

$$\text{Prob}(\sup_{t > t_0} \|X(t)\| > \epsilon) < \rho$$

In essence, this definition allows for the possibility that a subset of nonzero probability of the ensemble, $\{X(t)\}$ will not have $\|X(t)\| \leq \epsilon$ for all $t \geq t_0$, but that the probability that $\|X(t)\| > \epsilon$ can be made as small as desired by proper choice of $\|X(t_0)\|$.

An alternative class of definitions entails examination of ensemble averages only:

Definition 1 - M. The origin is stable in kth moment (i.m.k.) if and only if to any $\epsilon > 0$ there corresponds a $\delta(\epsilon)$ such that if $\|X(t_0)\| < \delta$

$$\sup_{t > t_0} E(\|X(t)\|^k) < \epsilon^k.$$

Definition 2 - M. The origin is asymptotically stable in kth moment if and only if it is stable i.m.k, and if for any $\mu > 0$ there exists a time $t_\mu$ such that

$$\sup_{t > t_\mu} E(\|X(t)\|^k) < \mu^k.$$

Since stability (asymptotic stability) with probability one implies a particular property for almost every member $X(t)$ of the output ensemble, while stability in moment implies a property of the ensemble average, one type of stability does not imply the other.

A third class of definitions is stability (asymptotic stability) in probability:

Definition 1 - P. The origin of the system is stable in probability (i.p.) if and only if for only $\epsilon > 0$ and $0 < \rho < 1$, there exists a $\delta(\epsilon, \rho)$ such that if $\|X(t_0)\| < \delta$ then

$$\sup_{t > t_0} \text{Prob}(\|X(t)\| > \epsilon) < \rho.$$

* This concept is called "stability in probability" by Khas'minskii. W.M. Wonham has suggested that to avoid the confusion between definitions 1 - AS' and 1 - P the former be called "strongly stable in probability" and the latter "weakly stable in probability." It might be pointed out that the definitions of stochastic stability have not as yet been crystallized.

Definition 2 - P. The origin of the system is asymptotically stable in probability, if and only if it is stable i.p. and if for any $\mu > 0$, $0 < \rho < 1$, there exists a $t_\mu$ such that

$$\sup_{t>t_\mu} \text{Prob}\,(\|X(t)\| > \mu) < \rho .$$

Asymptotic stability in probability is weaker than either asymptotic stability with probability one or asymptotic stability in moment, since either implies the first, but neither is implied thereby. The significance of the interchange of Prob and sup from definition 1-AS and 2-AS to 1-P and 2-P is that it is possible for every member of the ensemble $\{X(t)\}$ to violate the condition $\sup_t \|X(t)\| < \epsilon$ at a different time, yet at any fixed time instant only a denomerable subset violates the condition and thus $\sup_t \text{Prob}\,(\|X(t)\| > \epsilon) = 0$.

For visualization, suppose the output "ensemble" comprises 1000 functions $X_\iota(t)$ such that, at any given time, at most one of them has magnitude $|X(t)| > \epsilon$ and each of the functions $X_\iota(t)$, $\iota = 1, \ldots, 1000$, violates the condition $\sup_t |X(t)| < \epsilon$ at some time $t$. Thus $\text{Prob}\,(\sup_t |X(t)| > \epsilon) = 1$ yet at any fixed time $\text{Prob}\,(|X(t)| > \epsilon) < 1/1000$ and hence $\sup_t \text{Prob}\,(|X(t)| > \epsilon) = 1/1000$. As the number of members of the ensemble increases without bound we will have stability in probability but not stability with probability one.

The preceding conditions were for local asymptotic stability and need hold only for initial states in some closed region containing the origin. The origin of a system is called asymptotically stable in the large or globally asymptotically stable in any of the preceding senses if it is asymptotically stable for every finite $X(t_0)$.

## 5.2 Methods for Determining Lifetime

By use of a stochastic version of Lyapunov's second method Kushner [7] and Khas'minskii [8] have obtained sufficient conditions for stability with probability one for a class of randomly-excited systems. These conditions, however, can be satisfied only by those systems for which the effect of the excitation vanishes as the state approaches the origin. Unfortunately, this rules out many systems of practical interest. This limitation arises because for systems excited by white noise the confinement probability (3.49) tends to zero as $t \to \infty$, for any finite $N$ and any $y$, when the effect of the excitation does not vanish as the state approaches the origin. This can be demonstrated for system (4.2) as follows:

Assume that system (4.2) is excited by white noise with a normalized covariance matrix of unity, then $q(t|y)$ satisfies (4.12),

$$\frac{\partial q(t|y)}{\partial t} = \mathcal{L}_y [q(t|y)]. \tag{5.1}$$

The boundary condition for (5.1) is

$$q(t|y) \equiv 0 \qquad \text{for } t \geq 0, \ y \in \partial N \tag{5.2}$$

where $\partial N$ denotes the boundary of $N$, since, if the initial state is on or outside the boundary of the region the probability of remaining inside the region is obviously zero. The initial condition for (5.1) is

$$q(0|y) = 1 \text{ for } y \in N \tag{5.3}$$

since, if the initial state is inside the region, the probability of going to a state outside the region in no time at all $(t = 0)$ is zero.

Now suppose that $q(t|y)$ approaches a steady-state limiting function $q(y)$ as $t \to \infty$. Then, since $q(y)$ is independent of $t$, it must satisfy

$$\mathcal{L}_y(q) = g \cdot \nabla q + 1/2 \ \nabla \cdot D \nabla q = 0 \tag{5.4}$$

subject to the same boundary condition (5.2). It is not too difficult to show that $q(y) \equiv 0$ is the only solution to (5.4) which satisfies the boundary condition (5.2). First note that (5.4) can be expressed as

$$\mathcal{L}_y [q(y)] = g(y) \cdot \nabla q(y) + 1/2 \text{ trace } [D \, J(y)] = 0 \tag{5.5}$$

where $J(y)$ is the Jacobian matrix of $\nabla q(y)$. We note that if $q$ achieves a maximum at any point $y_0$ then $\nabla q(y_0) = 0$ and $J(y_0)$ is negative semi-definite. Since $D$ is positive semi-definite, the trace $DJ(y_0) < 0$. Hence $\mathcal{L}_y [q(y_0)] < 0$. This result is called the principal of the maximum [21] which can be used to show that $u(y) = 0$ is the only solution of

$$\mathcal{L}_y [u(y)] = c(y) \, u(y) \tag{5.6}$$

for any $c(y) > 0$ throughout a region $R$ and any $u(y)$ which satisfies the boundary condition $u(y) = 0$ on the boundary of $R$. If $u$ reaches a positive maximum at a point $y_0$ in $R$, then from (5.6), $\mathcal{L}_y [u(y_0)] = c(y_0)\, u(y_0) > 0$, which contradicts the principle of the maximum. Similarly if $u(y_0)$ has a negative minimum (i.e., $-u(y_0)$ has a positive maximum) at $y_0$ then $\mathcal{L}_y [-u(y_0)] = -c(y_0)\, u(y_0) > 0$, which again is a contradiction. Since $u(y) \equiv 0$ on the boundary and can have neither a positive maximum nor a negative minimum inside $R$, then $u(y)$ must be identically zero throughout $R$. To apply this result to $q(y)$, let

$$q(y) = u(y)\, [K - e^{-\alpha y_k}] \tag{5.7}$$

where $y_k$ is the kth component of $y$ and $u(y) = 0$ for $y$ on the boundary of $R$. Substitution of this expression into (5.5) leads to

$$\mathcal{L}_y [u] = g(y) \cdot \nabla\, u + 1/2 \text{ trace } [D J_u(y)] = c(y)\, u(y) \tag{5.8}$$

where $J(y)$ is the Jacobian matrix of $\nabla\, u(y)$ with respect to $y$ and

$$c(y) = \alpha \frac{g_k(y) + \alpha d_{kk}}{Ke^{-\alpha y_k} - 1} \tag{5.9}$$

where $d_{kk}$ is the kth diagonal element of $D$. Now, if $G$ is not the null matrix, then $d_{kk} > 0$ for at least one value of $k$. For such a $k$, it is always possible to choose $K$ and $\alpha$ such that $c(y) > 0$ throughout $R$. [For example, take $\alpha > \min_{y \epsilon R} g_k (y)/d_{kk}$ and $K > \exp\, [\max_{y \epsilon R} | y_k | )$. ] Hence, by the previous result, $u(y) \equiv 0$ is the only solution to (5.8) which satisfies the boundary condition $u(y) = 0$ on the boundary of $R$, and thus $q(y) \equiv 0$ is the only solution to (5.4) which is zero on the boundary $\partial N$. This means that the state ultimately escapes from any finite region of the origin, even if it starts at the origin itself! The natural distinction between a stable and an unstable system which can be made in the deterministic case is thus not as clear in the stochastic case.

The fact that $D$ is a constant matrix, and hence that $d_{kk} > 0$ (for some $k$) is needed for the construction of $c(y)$ of (5.8). This demonstration is not valid when $D$ may depend on $y$, and for appropriate dependence of $D$ on $y$, (5.4) may have a nonzero solution which satisfies the boundary conditions. Although in Section 4 only the case in which $D$ is constant was treated, it can be shown [20] that the backward equation (4.12) remains valid when D depends on y provided that $\mathcal{L}_y$ is defined by

$$\mathcal{L}_y u = f \cdot \nabla_y u + 1/2 \, \nabla_y [GG' \nabla_y u] \tag{5.10}$$

where

$$f = \begin{cases} g + \sum_{i=1}^{n} \dfrac{\partial d_i}{\partial y_i} & \text{for Ito's stochastic integral (4.5a)} \\ g + \sum_{i=1}^{n} \dfrac{\partial d_i}{\partial y_i} + \sum_{i=1}^{n} \sum_{k=1}^{n} \dfrac{\partial G_{ij}}{\partial y^k} G_{kj} & \text{for Stratonovich's stochastic integral (4.5b)} \end{cases}$$

where $d_i$ is the $i^{th}$ column of D. Thus in situations in which G depends on y it is possible to obtain a nontrivial asymptotic solution to (5.1) and hence it may be possible that $q(y) > 0$ and, in fact, this probability may arbitrarily close to unity when $G(y) \to 0$ with y. (Kushner [7] has considered this situation in some detail.)

In systems with G = const it is of practical interest to determine how fast the confinement probability $q(t|y)$ approaches zero as $t \to \infty$. In particular, one can define the half-life $t_h$ of the process by

$$q(t_h | 0) = 1/2 \tag{5.11}$$

which is the time it takes for the confinement probability of the process (which starts at the origin) to diminish to 1/2. The relative stability of systems can be compared by comparing the respective half-lives.

Another measure of system stability is the <u>average first passage time</u> to the boundary of a set. Evidently

$$q(t|y) = 1 - \text{Prob}\,[X(s) \notin N \text{ for some } s \in [0,t] \,|\, X(0) = y\,]$$

Let the random variable $T(y)$ denote the first passage time to the boundary starting at the state y. Let the corresponding density and distribution functions be, respectively, $r(t|y)$ and $R(t|y)$. (These are assumed to be independent of the starting time.)

Then

$$q(t|y) = 1 - R(t|y) \tag{5.12}$$

and the Laplace transform of (5.12) yields

$$\tilde{q}(s|y) = 1/s - \tilde{R}(s|y) \tag{5.13}$$

where $\sim$ indicates the Laplace transform. The average first passage time, defined as

$$\bar{T}(y) = \lim_{t\to\infty} \int_0^t \tau r(\tau|y)\, d\tau = \lim_{s\to 0} s\, \mathcal{L}\{ \int_0^t \tau r(\tau\ y)\, d\tau \} \tag{5.14}$$

can be obtained from $\tilde{r}(s|\ y)$ as follows:

Since $\mathcal{L}\{ \int_0^t \tau r(\tau|y)\, d\tau \} = \frac{1}{s}\, \mathcal{L}\ \{\tau r(\tau|y)\} = -\frac{1}{s}\frac{d}{ds}\tilde{r}\ (s|y)$

we see that

$$\bar{T}(y) = \lim_{s\to 0} - \frac{d}{ds}\, [\tilde{r}(s|y)] \tag{5.15}$$

The average first passage time $\bar{T}(y)$ can be related to the confinement probability $q(t|y)$.

Since $R(t|y) = \text{Prob}\ \{ T(y) < t\}$ (5.16)

we have

$$q(t|y) = 1 - R(t|y) \tag{5.17}$$

Taking the Laplace transform of (5.17) and using the fact that $\tilde{r}(s|y) = s\tilde{R}(s|y)$ in (5.15) gives

$$\bar{T}(y) = \lim_{s\to 0} \frac{d}{ds}\, [s\tilde{q}(s|y)] \tag{5.18}$$

Hence, since $q(\infty|y) = 0$,

$$\bar{T}(y) = -\lim_{t\to\infty} \int_0^t \tau \frac{dq}{d\tau}\, d\tau = \int_0^\infty q(\tau|y)\, d\tau \tag{5.19}$$

relates the mean-passage-time-to-the-boundary to q.

The average first passage time may also be calculated as the solution to a partial differential equation. Write the Laplace transform

$$\tilde{r}(s|y) = \int_0^\infty [\, 1 - st + \frac{s^2 t^2}{2!} - \ldots ]\, r(t|y)\, dt$$

Then the Laplace transform of $\partial r\,(t|\,y)/\partial t$ is

$$\int_0^\infty [\, s - s^2 t + \frac{s^3 t^2}{2!} - \cdots ]\, r(t|\,y)\, dt$$

Now take the Laplace transform of (5.2) and equate the coefficients of $s^n$. Let $\bar{T}^{(n)}(y)$ denote the nth moment of $\bar{T}(y)$. Then we have

$$- n\bar{T}^{(n-1)} = 1/2\ \nabla_y \cdot [\, GG'\ \nabla_y T^{(n)} ] + g(y) \cdot \nabla_y T^{(n)} \tag{5.20}$$

Therefore, the average lifetime can be calculated as the solution to (5.20) with $n = 1$, i.e. with $\bar{T}^{(1)} = \bar{T}$,

$$- 1 = 1/2\ \nabla_y \cdot [\, GG'\ \nabla_y \bar{T}\, ] + g(y) \cdot \nabla_y \bar{T} \tag{5.21}$$

with the boundary condition $\bar{T}(y) = 0$ for $y$ on the boundary of $N$.

## 6. ESTIMATES OF STABILITY

### 6.1 Analytical Methods

Exact determination of system lifetime necessitates solving (5.1) subject to (5.2) and (5.3). In most practical problems an exact solution to the partial differential equation cannot be obtained. Moreover, the parameters of the actual disturbances are usually not known with the precision that an exact solution warrants.

Thus there is a need for methods of obtaining lifetime estimates which do not entail solving the exact problem.

Upper Bounds - Upper bounds are useful to eliminate unsatisfactory system designs: if a particular choice of system parameters leads to an upper bound on $q$ which does not satisfy the design requirements, then certainly the exact solution for $q$ will also fail to do so.

An upper bound which is easy to calculate in many problems is

$$\hat{q}(t|y) = \text{Prob}\ [X(t) \in N \,|\, X(0) = y\ ] \tag{6.1}$$

Clearly, $q(t|y) \leq \hat{q}(t|y)$ because the set of sample functions for which $X(t) \in N$ at time $t$ contains the set of functions for which $X(s) \in N$ for all $s \in [0, t]$. (The first set contains more functions since it includes those functions which are not in $N$ at earlier times $s < t$.) Note that

$$\hat{q}(t|y) = \int_N p(x, t|y)\, dx \tag{6.2}$$

where $p(x, t|y)$ is the conditional probability density function.

For a linear system $p(x, t|y)$ is the normal density function given by (4.8). The evaluation of (6.3) can usually be accomplished by appropriate changes of variable and use of a table of error functions. (If necessary, the region $N$ can be inscribed in a larger region which makes the limits of integration on (6.2) more convenient.)

A similar device can be used to obtain another upper bound on $q(t|y)$ . If we can find a solution $q_R(t\,|\,y)$ to (5.1) subject to conditions (5.2) and (5.3) on a region R containing the given region N, then $q_R(t|y)$ is also an upper bound on $q(t|y)$. Thus we attempt to solve the backward equation subject to more convenient boundary conditions.

Lower Bounds - While an upper bound on q serves to weed out unsatisfactory designs, it cannot be used as an estimate of actual system performance; for this purpose, lower bounds are required. Two general methods have been described in the literature. One technique is based on a theorem due to Wonham [10]:

Let $V(y)$ be a twice continuously differentiable function such that

$$V(y) = 1 \qquad y \epsilon\ \partial N \tag{6.3}$$

and

$$0 \le V(y) \le 1 \ , \qquad y \epsilon N$$

If there exists a constant $\alpha > 0$ such that

$$\mathcal{L}[V(y)] \le \alpha\ [1 - V(y)] \quad , \quad y \epsilon\ N \tag{6.4}$$

then

$$q(t|y) \ge [1 - V(y)]\ e^{-\alpha t} \quad , \ t \ge 0 \tag{6.5}$$

where $\mathcal{L}$ is the backward operator defined in (3.28)*.

The problem of obtaining a lower bound can be reduced to finding the principal eigenvalue and eigenfunction of the boundary-value problem (6.3), (6.4). When $\mathcal{L}+\alpha$ is "self-adjoint" then an upper bound on the principal eigenvalue $\alpha_1$ , can be obtained by Dirichlet's principle [21].

A second technique due to Kushner [9] is based on the concept of a stochastic Lyapunov function [7] and on theorems of Doob [12] and Dynkin [4]. A lower bound on q can be obtained using the following corollary to a more general theorem established in Reference [9]:

* The subscript y on $\mathcal{L}$ is omitted when $\mathcal{L}$ operates on a function of only a single space variable.

Let $V(y)$ be a continuous non-negative function, which is bounded and has continuous and bounded second derivatives in the set $Q = \{y: V(y) \le 1\}$. For system (4.9) (where $g(y)$ satisfies a uniform Lipschitz condition in $Q$ and is bounded by $K(1+ ||y||^2)^{1/2}$) if there is a non-negative, integrable function $\varphi(s)$ on the interval $0 \le s \le t \le \infty$, such that

$$\mathcal{L}[V(y)] \le \varphi(s) \quad \text{for all} \quad y \in Q \tag{6.6}$$

then

$$\mathrm{Prob}\{\sup_{0\le s\le t} V(X(s)) < 1 \mid X(0) = y\} \ge 1 - V(y) - \Phi(t) \tag{6.7}$$

where
$$\Phi(t) = \int_0^t \varphi(s)\, ds \tag{6.8}$$

and $\mathcal{L}$ is the backward operator defined in (3.28).

If $V(y)$ is selected such that $Q$ is a subset of $N$, then $q(t|y)$ is greater than or equal to the left-hand side of (6.7). Thus the right-hand side of (6.7) gives a lower bound to $q$. The major difficulty with this approach is that in cases for which the right-hand side of (6.7) becomes less than zero after some finite time $t'$ (6.7) gives an unduly pessimistic lower bound. However, as in the case of deterministic systems, a better choice of $V(y)$ and $\varphi(s)$ may lead to a better estimate. The bound (6.5) does not have this shortcoming since, for finite time, the right-hand side of (6.5) will always be greater than zero.

A third approach for obtaining a lower bound on $q(t|y)$ is similar to one described for upper bounds. If a solution $q_R(t|y)$ can be found to (5.1) subject to (5.2) and (5.3) on a region $R$ contained in $N$ over the time interval $[0, t]$, then $q_R(t|y)$, the probability of remaining in the smaller region $R$, is certainly a lower bound on the confinement probability $q(t|y)$. Note that since the transition density $p(x, t|y)$ and the conditional characteristic function $C(\nu, t|y)$ are known to satisfy the backward equation. These are useful candidate functions in applying this technique as well as the corresponding one for upper bounds.

Another approach, that can be used for obtaining a lower bound on $q(t\,|\,0)$, makes direct use of an inequality to Dynkin [4]:

$$q(t\,|0) \geq 1 - \frac{t}{\bar{T}(0)} \tag{6.9}$$

where $\bar{T}(y)$ is found by solving (5.21). In problems for which (5.21) is easier to solve than (5.1), the bound (6.9) will be more easily obtained than the exact solution to the problem. Since the right-hand side of (6.9) becomes zero when $t = \bar{T}(0)$, (6.9) may give an unduly pessimistic lower bound for $q$.

## 6.2 Numerical Methods

Monte-Carlo simulation is a "brute-force" method of obtaining an approximate solution for $q(t\,|y)$. In this approach the equation (4.4) is replaced by the difference equation

$$x_{n+1} = x_n + g(x_n) + v_n \tag{6.10}$$

where

$$g(x_n) = \int_{n\Delta t}^{(n+1)\Delta t} g(x)\, dt \tag{6.11}$$

$$v_n = \int_{n\Delta t}^{(n+1)\Delta t} G\, dw \tag{6.12}$$

Random vectors $v_n$ are generated statistically and applied to (6.10). An ensemble of trajectories is computed and the ratio

$$r(n\Delta t\,|\,y) = \frac{\text{Number of trajectories for which } x_n \in N\text{, starting with } x_0 = y}{\text{Total number of trajectories}}$$

is used as an approximation to $q(n\Delta t\,|y)$.

Another numerical technique which may be used in obtaining solutions to (5.1) is Galerkin's method. Briefly, for a scalar process we assume a solution in the form

$$q(t|y) = \sum_{n=1}^{\infty} f_n(y) q_n(t) \tag{6.13}$$

where $f_n(y)$ are appropriately chosen orthogonal functions satisfying the boundary conditions. Upon substituting (6.13) into (5.1) we obtain an infinite number of ordinary differential equations for the $q_n(t)$ 's subject to the given initial condition. The numerical technique consists of truncating the infinite set of equations after N terms, and then integrating the resulting Nth order system by standard numerical techniques. The circumstances under which the solution to the Nth order system is a valid approximation to the infinite order system is discussed in Reference [22] and in other references concerning Galerkin's method.

## 7. EXAMPLES

### 7.1 Integrator with White Noise Input (Wiener Process)

Consider the first-order system

$$\dot{x} = \xi \tag{7.1}$$

where $\xi$ is white noise with variance $\sigma^2 \delta(t)$. In Section 4 we demonstrated that the output process $\{x\}$ is a Wiener process and thus the confinement probability satisfies the diffusion equation (3.31).

$$\frac{\partial q}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 q}{\partial y^2} \tag{7.2}$$

with boundary conditions

$$q(t|\pm B) = 0 \qquad \text{for all } t \tag{7.3}$$

and initial condition

$$q(t|0) = \begin{cases} 1, & |y| < B \\ 0 & |y| \geq B \end{cases} \tag{7.4}$$

Take the Laplace transform of (7.2) and apply initial condition (7.4) to obtain

$$\frac{\sigma^2}{2} \frac{d^2\tilde{q}}{dy^2} - s\tilde{q} = -1 \tag{7.5}$$

where $\tilde{q} = \tilde{q}(s|y)$ is the Laplace transform of $q(t|y)$. A particular solution to (7.5) is clearly $\tilde{q} = 1/s$. Two linearly independent solutions to the homogeneous form of (7.5) are $\cosh(y\sqrt{2s}/\sigma)$ and $\sinh(y\sqrt{2s}/\sigma)$. Thus

$$\tilde{q}(s|y) = \frac{1}{s} + A_1 \cosh\left(\frac{y}{\sigma}\sqrt{2s}\right) + A_2 \sinh\left(\frac{y}{\sigma}\sqrt{2s}\right) \tag{7.6}$$

where $A_1$ and $A_2$ are constants to be determined from boundary conditions (7.3). Substitution of (7.6) into (7.3) yields

$$\tilde{q}(s|y) = \frac{1}{s}\left(1 - \frac{\cosh \frac{y}{\sigma}\sqrt{2s}}{\cosh \frac{B}{\sigma}\sqrt{2s}}\right) \tag{7.7}$$

The well-known inverse Laplace transform of (7.7) is

$$q(t|y) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left[-\frac{(2k+1)^2 \pi^2 \sigma^2 t}{8B^2}\right] \cos\left[(2k+1)\frac{\pi}{2}\frac{y}{B}\right] \tag{7.8}$$

and is plotted in Figure 7-1 (adapted from Chart 2 of [23]).

An alternate form of $q(t|y)$ can be obtained by first expanding (7.7) in the infinite series

$$\tilde{q}(s|y) = \frac{1}{s}\left\{1 - \sum_{k=0}^{\infty} (-1)^k \left[\exp\left(-\frac{\sqrt{2s}}{\sigma}\right)(2k+1)B - y\right] + \left[\exp -\frac{\sqrt{2s}}{\sigma}(2k+1)B + y\right]\right\}$$

Then, taking the inverse transform term by term gives

$$q(t|y) = 1 - \sum_{k=0}^{\infty} (-1)^k \left\{\operatorname{erfc}\left[\frac{(2k+1)B - y}{\sigma\sqrt{2t}}\right] + \operatorname{erfc}\left[\frac{(2k+1)B + y}{\sigma\sqrt{2t}}\right]\right\} \tag{7.9}$$

which can be interpreted as

$$q(t|y) = 1 - 2\sum_{k=0}^{\infty} (-1)^k \left[1 - \text{Prob}\left\{|X(t)| < (2k+1)B \,|\, X(0) = y\right\}\right] \tag{7.10}$$

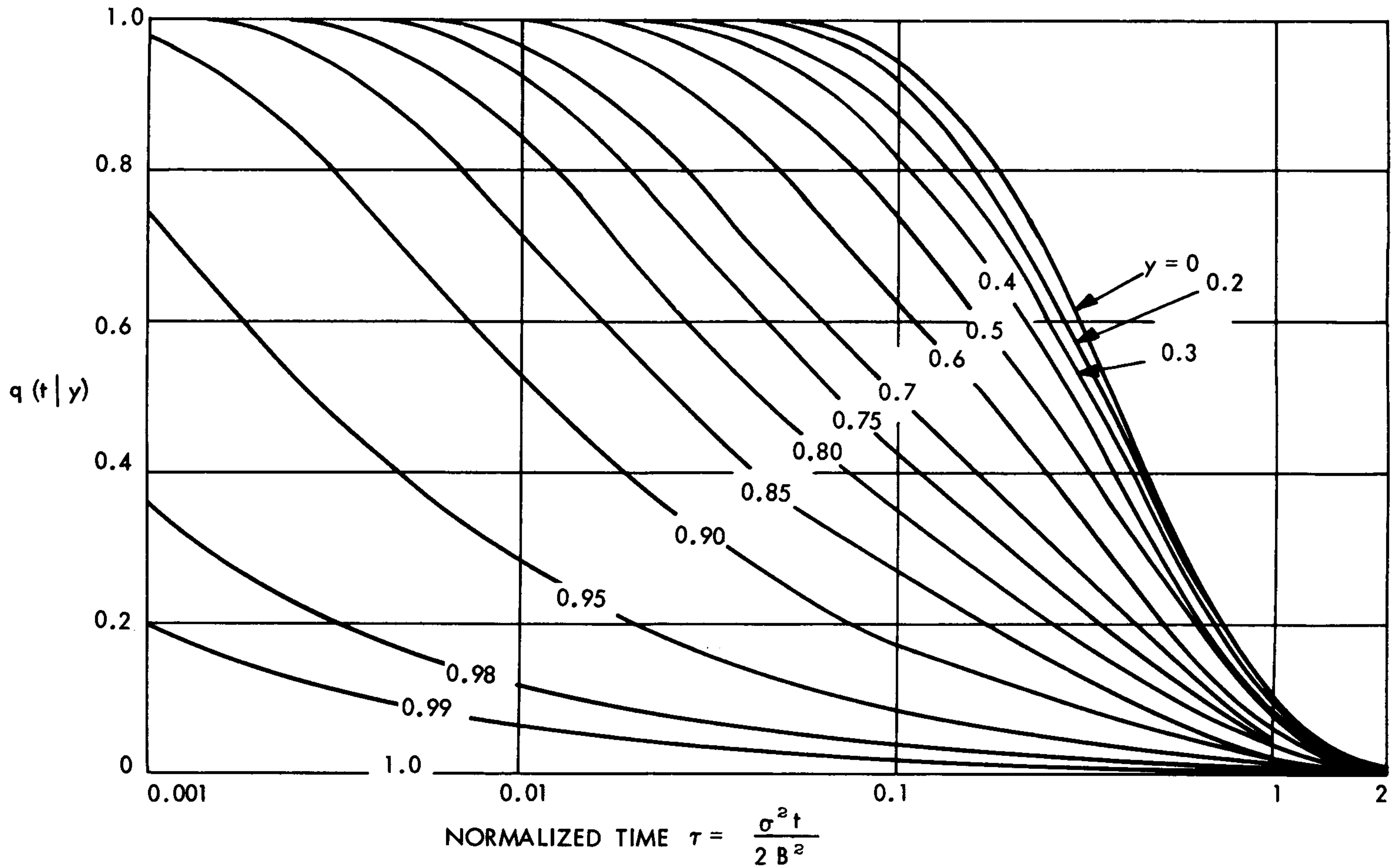
By using definition (5.17) and $\tilde{q}(s|0)$ from (7.7) we find that the average first passage time to the boundary given the initial state $y = 0$ is

$$\bar{T}(0) = \frac{B^2}{\sigma^2} \tag{7.11}$$

Thus, as one intuitively expects, the average first passage time to the boundary increases with the size of the region and decreases with the magnitude $\sigma^2$ of the disturbance.

**FIGURE 7-1**

**CONFINEMENT PROBABILITY OF INTEGRATOR OUTPUT**



The upper bound (6.2) on the confinement probability is given by

$$\hat{q}(t|0) = \int_{-B}^{B} p(x, t|0)\, dx \tag{7.12}$$

where, as determined in (2.10),

$$p(x, t|0) = \frac{1}{\sqrt{2\pi}\,\sigma\, t^{\frac{1}{2}}} \exp[-x^2/2\sigma^2 t]$$

The upper bound (7.12) is given by

$$\hat{q}(t|0) = \operatorname{erf}\left[B/\sqrt{2\sigma^2 t}\right] \tag{7.13}$$

Wonham's lower bound (6.5) is obtained by choosing $V(y) = 1 - \cos\left(\frac{\pi y}{2B}\right)$. Thus $\alpha = \frac{\sigma^2 \pi^2}{8B^2}$ satisfies (6.4) with equality and (6.5) becomes

$$q(t|0) \geq \exp[-\pi^2 \sigma^2 t/8B^2] \tag{7.14}$$

A lower bound using Kushner's method may be obtained by selecting

$$V(y) = y^2/B^2 \tag{7.15}$$

Then

$$\mathcal{L}[V(y)] = \sigma^2/B^2$$

where $\mathcal{L}$ is the right-hand side of (4.17). Thus

$$\varphi(t) = \sigma^2/B^2$$

and (6.7) yields the estimate

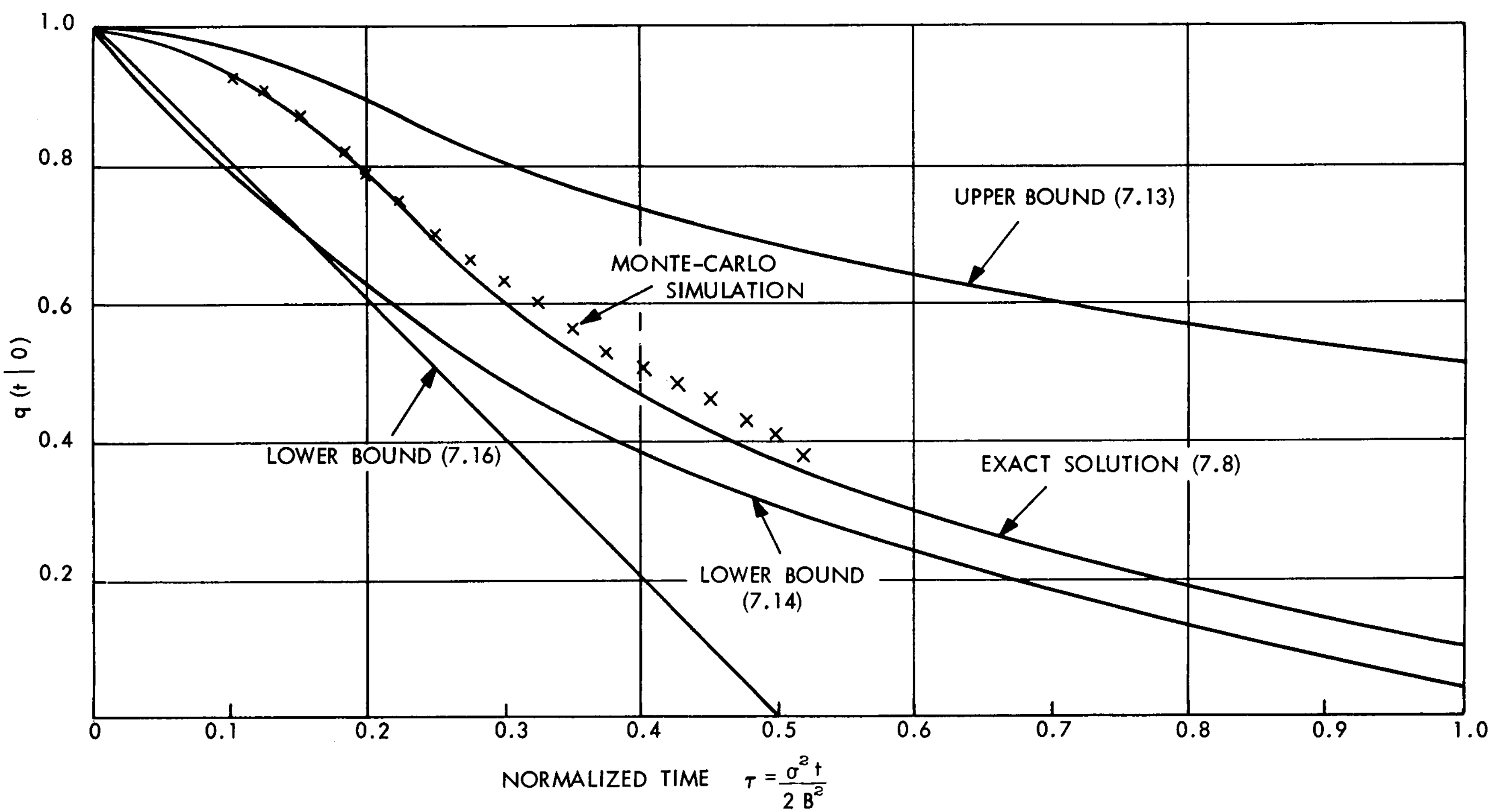
$$q(t|0) > 1 - (\sigma^2 t/B^2) \tag{7.16}$$

Although another choice of V-function might yield a better bound than (7.16), the choice (7.15) suffices to illustrate the method.

Another lower bound on $q(t|0)$ can be obtained as follows: Let $p_l(t) = \hat{q}(t|0)$ and let $p_0(t) = 1 - p_l(t)$.

**FIGURE 7-2**

**INTEGRATOR CONFINEMENT PROBABILITY WITH WHITE NOISE INPUT**



Then by using only the first term in the bracket of (7.10) and the fact that

$$\text{Prob } \{|X| < B\} < \text{Prob } \{|X| < 3B\} < \text{Prob } \{|X| < 5B\} < \ldots,$$

we have

$$q(t|0) > -1 + 2 \text{ Prob } \{|X| < B\}$$

$$q(t|0) > \text{Prob } \{|X| < B\} - \text{Prob } \{|X| > B\} = p_i - p_0 \tag{7.17}$$

Note that $p_i - p_0$ is also a lower bound for $q(t|y)$ for any $|y| < B$.

Dynkin's lower bound (6.9) can be obtained by using (7.11) to give

$$q(t|0) > 1 - \frac{\sigma^2 t}{B^2} \tag{7.18}$$

which coincidentally is identical with (7.16)

A Monte-Carlo simulation in accordance with the method described above was also performed. Figure 7-2 contains graphs of the exact solution (7.8), estimates (7.13), (7.14), and (7.16), and the Monte-Carlo simulation as function of normalized time. From the figure we see that: (a) the estimate (7.16) becomes zero for finite time and thus is not as useful as (7.14), (b) the Monte-Carlo simulation agrees well with the exact solution, (c) the upper bound (7.13) becomes a rather poor estimate as time increases. The lower bound $p_i - p_0$ is not shown in Figure 2 since for the scale used in the Figure $p_i - p_0$ is almost identical with the exact solution. The $p_i - p_0$ technique will be discussed further in Section 9.

From Figure 7-2 it is seen that the half-life is given by

$$t_h = 0.75B^2/\sigma^2 \tag{7.19}$$

## 7.2 Integrator with Poisson Impulse Input (Poisson Step Process)

Consider the system (7.1) where $\xi$ is a Poisson impulse process. From (3.41) the confinement probability satisfies

$$\frac{\partial q}{\partial t} = - \lambda \left[ q - \int_{-\infty}^{\infty} q(t|\, y - z)\, \gamma_1(z)\, dz \right] \tag{7.20}$$

where $\lambda$ is the average number input impulses per unit time and $\gamma_1(\xi)$ is the density function for the magnitude of each step. The boundary conditions and initial condition on (7.20) are given by (7.3) and (7.4).

A solution to (7.20) shall be constructed by assuming that the solution is separable and can be expanded in terms of orthogonal functions $\{\psi_k(y)\}$, i.e.,

$$q(t|\, y) = \sum_{k=1}^{\infty} \Phi_k(t)\, \psi_k(y) \tag{7.21}$$

where $\psi_k(B) = \psi_k(-B) = 0$

$$\Sigma_k \Phi_k(0)\, \psi_k(y) = 1$$

$$\int_{-B}^{B} \psi_k(y)\, \psi_j(y)\, dy = \delta_{kj}$$

Then substituting (7.27) into (7.20) gives

$$\Sigma_k \dot{\Phi}_k(t)\, \psi_k(y) = -\lambda \sum_k \Phi_k(t)\, \psi_k(y) + \lambda \int_{-\infty}^{\infty} \Sigma \Phi_k(t)\, \psi_k(y - z)\, \gamma_1(z)\, dz \tag{7.22}$$

Multiplying by $\psi_n(y)$ and integrating with respect to $y$ between $-B$ and $B$ results in

$$\dot{\Phi}_n(t) = -\lambda \Phi_n(t) + \lambda \int_{-B}^{+B} \psi_n(y) \int_{-\infty}^{\infty} \sum_k \Phi_k(t)\, \psi_k(y - z)\, \gamma_1(z)\, dz\, dy$$

Formal interchanging limits and summation gives

$$\dot{\Phi}_n(t) = -\lambda \Phi_n(t) + \lambda \sum_k \Phi_k(t) \int_{-\infty}^{\infty} \gamma_1(z) \int_{-B}^{B} \psi_n(y)\, \psi_k\, (y - z)\, dy\, dz$$

If we use the following set of orthogonal functions

$$\psi_k(y) = \frac{1}{\sqrt{B}} \cos \left[ \frac{(2k-1)\pi}{2B} y \right]$$

then

$$\dot{\Phi}_n(t) = -\lambda \Phi_n(t) + \lambda \sum_k \Phi_k \int_{-\infty}^{\infty} \gamma_1(\xi) \frac{1}{B} \int_{-B}^{B} \cos \left[ \frac{(2n-1)}{2B} y \right] \cos \left[ \frac{(2k-1)\pi}{2B} (y - z) \right] dy\, dz \tag{7.23}$$

Let

$$\gamma_{kn} = \frac{1}{B} \int_{-B}^{B} \cos \left[\frac{(2m-1)}{2B} y\right] \cos \left[\frac{(2k-1)\pi}{2B} (y - z)\right] dy$$

$$= \frac{1}{B} \cos \frac{(2k-1)\pi z}{2B} \int_{-B}^{B} \cos \frac{(2n-1)\pi y}{2B} \cos \frac{(2k-1)\pi y}{2B} dy$$

$$+ \frac{1}{B} \sin \frac{(2k-1)\pi z}{2B} \int_{-B}^{B} \cos \frac{(2m-1)\pi y}{2B} \sin \frac{(2k-1)\pi y}{2B} dy$$

Then

$$\gamma_{kn} = \begin{cases} 0 & k \neq n \\ \frac{1}{B} \cos\left(\frac{(2n-1)\pi}{2B} z\right) & k = n \end{cases}$$

and the differential equation (7.23) reduces to

$$\dot{\Phi}_n(t) = -\lambda \Phi_n(t) + \lambda \Phi_n \frac{1}{B} \int_{-\infty}^{\infty} \gamma_1 (z) \cos\left(\frac{(2n-1)\pi}{2B} z\right) dz \qquad n = 1, 2, \ldots \tag{7.24}$$

For any even density function the integral is equivalent to the moment generating function. In the case in which $\gamma_1$ (z) is Gaussian

$$\gamma_1(z) = \frac{1}{\sqrt{2\pi}\,\hat{\sigma}} e^{-z^2/2\hat{\sigma}^2}$$

then

$$\frac{1}{\sqrt{2\pi}\,\hat{\sigma}} \int_{-\infty}^{\infty} e^{-z^2/2\hat{\sigma}^2} \cos \frac{(2m-1)\pi}{2B} z \, dz = \exp\left(-\frac{1}{2}\left[\frac{2m-1)\pi\hat{\sigma}}{2B}\right]^2\right)$$

and the differential equation (7.24) becomes

$$\dot{\Phi}_n(t) = -\lambda\Phi_n\left[1 - \frac{1}{B}\exp\left(-\frac{1}{2}\left[\frac{(2m-1)\pi\hat{\sigma}}{2B}\right]^2\right)\right] \qquad n = 1, 2, \ldots$$

Since

$$1 = \sum_k \Phi_k(0)\,\frac{1}{\sqrt{B}}\cos\frac{(2k-1)\pi y}{2B} \qquad (7.25)$$

The coefficients $\Phi_k(0)$ can be determined by multiplying (7.25) by $\frac{1}{\sqrt{B}}\cos\frac{(2k-1)\pi y}{2B}$ and integrating with respect to $y$ between $-B$ and $+B$,

$$\Phi_k(0) = \frac{1}{B}\int_{-B}^{B}\cos\frac{(2k-1)\pi y}{2B}\,dy = (-1)^{k+1}\frac{4}{(2k-1)\pi}$$

The solution to the differential equation is then

$$\Phi_n(t) = (-1)^{k+1}\frac{4}{(2k-1)\pi}\exp\left\{-\lambda t\left[1 - \exp\left(-\frac{1}{2}\left(\frac{2m-1}{2B}\pi\hat{\sigma}\right)^2\right)\right]\right\}$$

It follows that the solution to (7.20) with $\gamma_1(\xi)$ Gaussian is

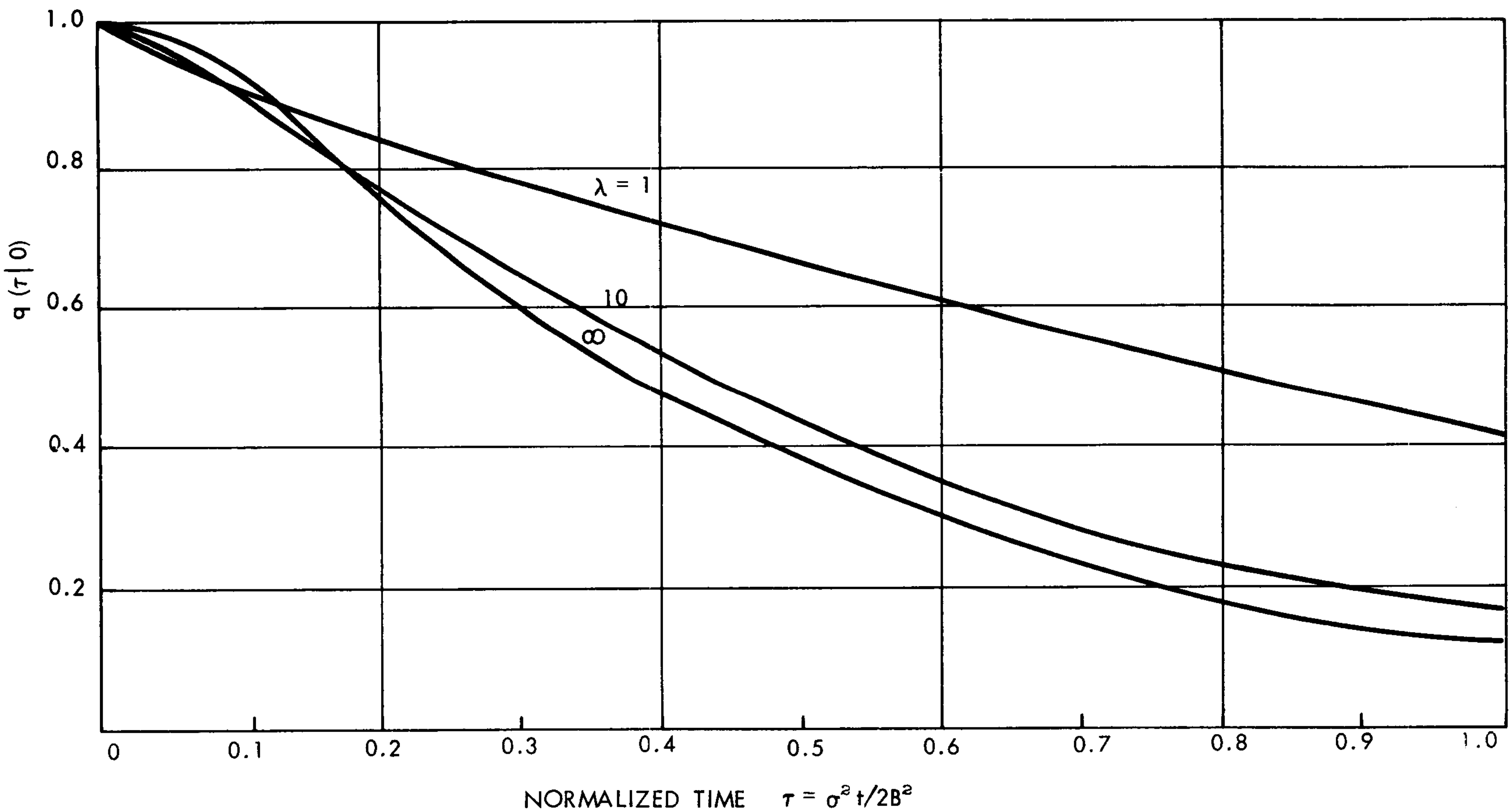
$$q(t|y) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)}\cos\left(\frac{2k-1}{2B}\right)\pi y \exp\left\{-\lambda t\left(1 - \exp\left[-\frac{1}{2}\left(\frac{2k-1}{2B}\right)^2\pi^2\hat{\sigma}^2\right]\right)\right\} \qquad (7.26)$$

If we allow $\lambda \to \infty$, keeping $\lambda\hat{\sigma}^2$ constant then the Poisson impulse process approaches a white noise process. The solution approaches

$$q(t|y) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)}\cos\frac{(2k-1)}{2B}\pi y\exp\left[-\left(\frac{t}{2}\right)\left(\frac{2k-1}{2B}\right)^2\pi^2\sigma^2\right]$$

where $\sigma^2 = \lambda\hat{\sigma}^2$ which is the result obtained in the case of a white noise input.

**FIGURE 7-3**
**INTEGRATOR CONFINEMENT PROBABILITY WITH POISSON IMPULSE PROCESS INPUT**



The probability (7.26) for an integrator excited by Poisson impulses is compared with an integrator excited by white noise in Figure 7-3. The variance of the Poisson process was adjusted so that it had the same second moment as the corresponding white noise process, i.e., $\lambda\hat{\sigma}^2 = \sigma^2$ where $\sigma^2$ is the variance of the equivalent white noise process.

Although the Poisson models are analytically less tractable than the corresponding Wiener process, or white noise limiting forms, it may be possible to obtain useful estimates of performance. In particular it would appear that the use of white noise with $\sigma^2 = \lambda\hat{\sigma}^2$ in place of Poisson pulses yields a conservative (i.e., pessimistic) estimate of system lifetime. In particular for a first-order diffusion process with white noise excitation, the half-life is given by

$$t_w = 0.75 \frac{B^2}{\sigma^2} = 0.75 \frac{B^2}{\lambda\hat{\sigma}^2}$$

as $\lambda \to \infty$ with $\lambda\hat{\sigma}^2 = \sigma^2 = \text{const.}$ With Poisson pulse excitation with $\lambda \to 0$ we have found that

$$t_p = \frac{0.69}{\lambda} \left[ 1 - \exp\left( - \frac{\pi^2 \hat{\sigma}^2}{8B^2} \right) \right]^{-1}$$

The ratio of these half-lives, is

$$\frac{t_w}{t_p} = 0.92 \frac{B^2}{\hat{\sigma}^2} \left[ 1 - \exp\left(- \frac{\pi^2 \hat{\sigma}^2}{8B^2}\right) \right] = 1.23 \; \Lambda \left(1 - e^{-0.92/\Lambda}\right) \qquad (7.27)$$

where $\Lambda = \lambda t_w = 0.75\, B^2/\hat{\sigma}^2$ is the number of arrivals per unit of the "white noise half-life". For $\Lambda = 1$, (i.e., for one arrival per unit white noise half-life, which would be regarded as quite small) (7.27) gives

$$t_p = 1.32 \; t_w$$

In other words, use of the white noise model underestimates the half-life by about 32%.

Since the white noise approximation tends to give a conservative estimate which is not entirely unrealistic, it would appear that the white noise approximation would be useful for $\Lambda$ even as low as unity.

It remains to be established whether use of the equivalent white noise model (i.e., $\lambda\hat{\sigma}^2 = \sigma^2$) is similarly conservative in higher order systems.

## 7.3 First-Order Linear System

Consider the system

$$\dot{x} = - \alpha x + \xi \tag{7.28}$$

where $\xi$ is white noise with variance $\sigma^2 \delta(t)$. From (4.16), the confinement probability satisfies the backward equation

$$\frac{\partial q}{\partial t} = -ay \frac{\partial q}{\partial y} + \frac{\sigma^2}{2} \frac{\partial^2 q}{\partial y^2} \tag{7.29}$$

subject to conditions (7.3) and (7.4). Take the Laplace transform of (7.29) to obtain

$$\frac{\sigma^2}{2} \frac{d^2\tilde{q}}{dy^2} - ay \frac{d\tilde{q}}{dy} - s\tilde{q} = -1 \tag{7.30}$$

As in example 7.1, a particular solution to (7.30) is clearly $\tilde{q} = \frac{1}{s}$. Introduce the transformation $z = ay^2/\sigma^2$. Then the homogeneous form of (7.30) becomes

$$z \frac{d^2\tilde{q}}{dz^2} + (\frac{1}{2} - z) \frac{d\tilde{q}}{dz} - \frac{s}{2a} \tilde{q} = 0 \tag{7.31}$$

which has linearly independent solutions

$$F_1(s|y) = M(\frac{s}{2a}, \frac{1}{2}, \frac{ay^2}{\sigma^2}) \tag{7.32}$$

$$F_2(s|y) = M(\frac{s}{2a}, \frac{3}{2}, \frac{ay^2}{\sigma^2})\sqrt{\frac{ay^2}{\sigma^2}} \tag{7.33}$$

where $M(a, b, x)$ is the "confluent hypergeometric function" [24]. Note that $F_1$ is an even function of $y$ and $F_2$ is an odd function of $y$. Reasoning as in example 7.1, we conclude that

$$\tilde{q}(s|y) = \frac{1}{s}\left(1 - \frac{M\left(\frac{s}{2a}, \frac{1}{2}, \frac{ay^2}{\sigma^2}\right)}{M\left(\frac{s}{2a}, \frac{1}{2}, \frac{aB^2}{\sigma^2}\right)}\right) \tag{7.34}$$

Again, following the procedure of example 7.1, the inverse Laplace transform of (7.34) gives the required solution:

$$q(t|y) = \sum_{k=1}^{\infty} c_k e^{-s_k t} \quad M\left(\frac{s_k}{2a}, \frac{1}{2}, \frac{ay^2}{\sigma^2}\right) \tag{7.35}$$

where the $s_k$ are the poles of $\tilde{q}(s|y)$, i.e., the zeros of $M\left(\frac{s}{2a}, \frac{1}{2}, \frac{aB^2}{\sigma^2}\right)$, and the coefficients $c_k$ are given by:

$$c_k = \frac{1}{s_k} \left. \frac{1}{\frac{d}{ds}\left[M\left(\frac{s}{2a}, \frac{1}{2}, \frac{aB^2}{\sigma^2}\right)\right]} \right|_{s = s_k}$$

The average first passage time for the linear system is obtained from (5.18), (7.34) and by using an infinite series for the hypergeometric functions [24]. This results in

$$\bar{T}(0) = \frac{1}{2a} \sum_{j=1}^{\infty} \frac{\left(\frac{aB^2}{\sigma^2}\right)^j}{j(1/2)_{(j)}} \tag{7.36}$$

where $(n)_{(j)} = n(n+1) \ldots (n+j-1)$. Figure 7-4 contains a graph of normalized average first passage time vs. normalized system gain $\alpha = aB^2/\sigma^2$ which reveals that average lifetime increases rapidly with the gain of the system.

A numerical solution to (7.29) can be obtained by Galerkin's method as follows: normalize (7.29) by the introduction of the following normalized quantities

$z = y/B$

$\tau = \sigma^2 t/2B^2$

in terms of which (7.29) becomes

$$\frac{\partial q}{\partial \tau} = -\alpha z \frac{\partial q}{\partial z} + \frac{\partial^2 q}{\partial z^2} \tag{7.37}$$

where $\alpha = \frac{2B^2}{\sigma^2} a$

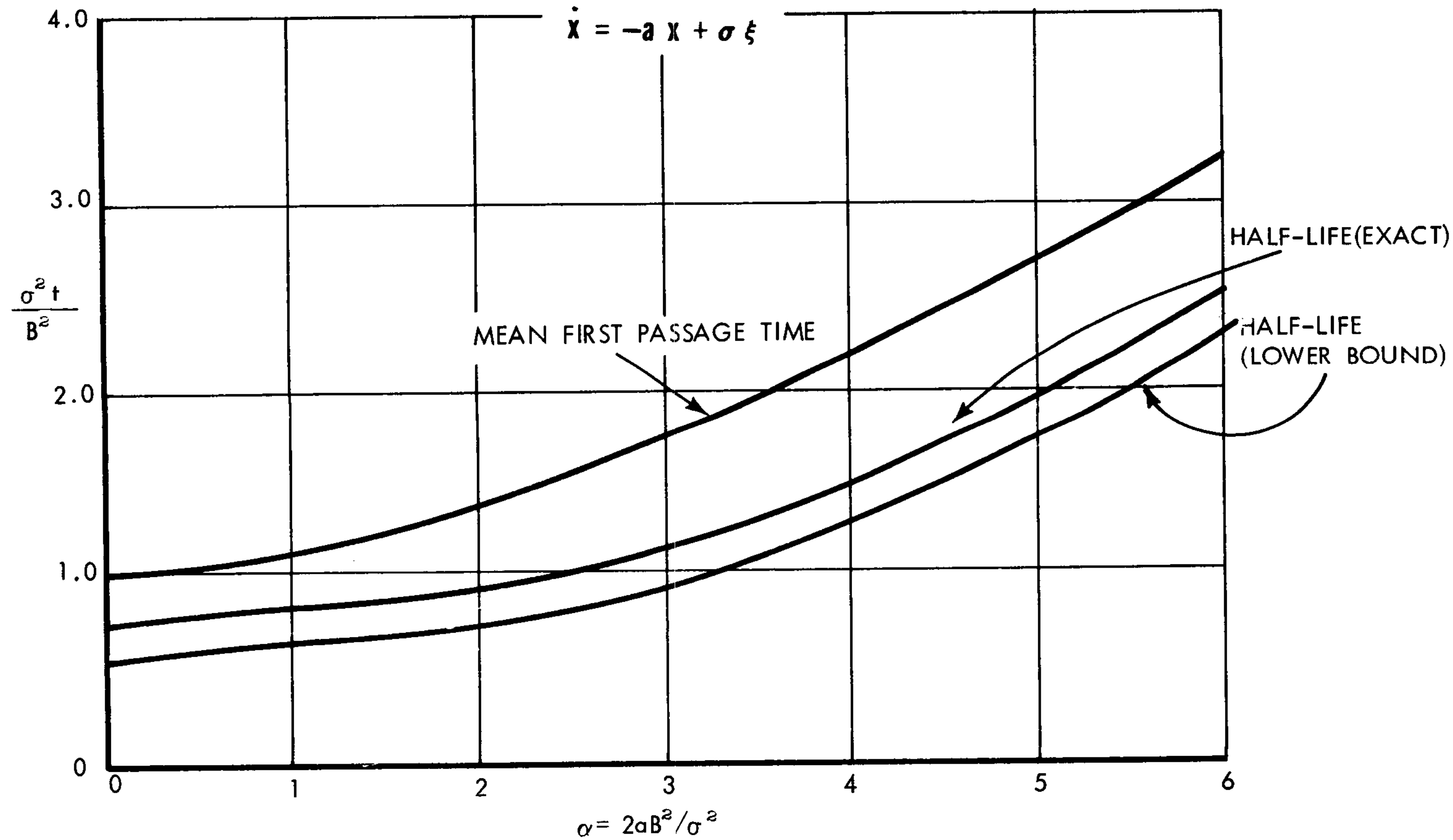
with initial and boundary conditions

$$q(0|z) = 1 \tag{7.38}$$

$$q(\tau| \pm 1) = 0 \text{ for all } \tau > 0 \tag{7.39}$$

FIGURE 7-4

AVERAGE FIRST PASSAGE TIME OF LINEAR SYSTEM



We assume a solution of (7.37) in the form

$$q(\tau | y) = \sum_{n=0}^{\infty} f_n(z)\, q_n(\tau) = \underline{f}'(z)\underline{q}(\tau) \tag{7.40}$$

where
$$\underline{f}' = [f_1, \ldots, f_n, \ldots] \tag{7.41}$$

$$\underline{q}' = [q_1, \ldots, q_n, \ldots] \tag{7.42}$$

are infinite dimensional vectors. An expansion of the form (7.40) is valid if the functions $f_1, f_2, \ldots$ constitute a complete set.

We also assume that the functions $f_1, f_2, \ldots,$ have the following properties

$$f_k(+1) = f_k(-1) = 0 \quad \text{all } k \tag{7.43}$$

$$\int_{-1}^{1} f_k(y) f_n(y)\, dy = c_k \delta_{kn} \tag{7.44}$$

where $\delta_{kn}$ is the Kronecker delta. Property (7.43) insures that (7.39) is satisfied by (7.40) and the "orthogonality condition" (7.44) permits the reduction of (7.37) to a system of infinitely many first-order ordinary differential equations. In particular, substitution of (7.40) into (7.37) gives

$$\sum_{n=1}^{\infty} f_n(z)\dot{q}_n(t) = \sum_{n=1}^{\infty} \left(-\alpha z \frac{\partial f_n}{\partial z} + \frac{\partial^2 f_n}{\partial z^2}\right) q_n \tag{7.45}$$

On multiplication of both sides of (7.45) by $f_k(z)$ and integration with respect to $z$ between the limits of -1 and +1 one obtains

$$\dot{q}_k = \frac{1}{c_k} \sum_{n=1}^{\infty} \left[ \int_{-1}^{1} f_k(z)\left(-\alpha z \frac{\partial f_n}{\partial z} + \frac{\partial^2 f_n}{\partial z^2}\right) dz \right] q_n \tag{7.46}$$

after use of (7.44). The linear system (7.46) can be represented in the vector-matrix form

$$\dot{\underline{q}} = C^{-1} [-\alpha M + L]\underline{q} \tag{7.47}$$

where $\underline{q}$ is the infinite-dimensional vector (7.42) and $C$, $M$, and $L$ are infinite dimensional matrices, with elements given by:

$$C_{kn} = c_k \delta_{kn} \tag{7.48}$$

$$M_{kn} = \int_{-1}^{1} z f_k(z) \frac{df_n}{dz} dz \tag{7.49}$$

$$L_{kn} = \int_{-1}^{1} f_k(z) \frac{d^2 f_n}{dz^2} dz \tag{7.50}$$

The initial condition $q(0)$ for (7.47) is obtained from (7.38):

$$q(0| z) = \sum_{n=0}^{\infty} f_n(z) q_n(0) = 1 \tag{7.51}$$

On multiplying (7.51) by $f_k(z)$, integrating with respect to $z$ between -1 and +1, and using the orthogonality condition (7.44) we obtain

$$q_k(0) = \frac{1}{c_k} \int_{-1}^{1} f_k(z)\, dz \tag{7.52}$$

The numerical technique consists of truncating the infinite matrices and vectors after $N$ terms, and then integrating the resulting $N\underline{th}$ order system (7.47) by standard numerical techniques. The circumstances under which the solution to the $N\underline{th}$ order system is a valid approximation to the infinite order system is discussed in various references concerning Galerkin's method [22].

Integration of (7.49) by parts results in:

$$M_{kn} = \int_{-1}^{1} y f_k \frac{df_n}{dy} dy = y f_k(y)\, f_n(y) \Big|_{-1}^{1} - \int_{-1}^{1} y f_n \frac{df_k}{dy} dy - \int_{-1}^{1} f_n(y)\, f_k(y)\, dy$$

which upon use of (7.43) and (7.44) becomes

$$M_{kn} = -M_{nk} - c_k \delta_{kn}$$

Thus

$$M_{kn} = -M_{nk} \qquad k \neq n$$

$$M_{kk} = -\frac{c_k}{2}$$

The matrix $M$ thus has the following structure

$$M = \begin{bmatrix} -\frac{c_1}{2} & M_{12} & M_{13} & \cdots \\ -M_{12} & -\frac{c_2}{2} & M_{23} & \cdots \\ -M_{13} & -M_{23} & -\frac{c_3}{2} & \cdots \\ & \cdots & & \\ & \cdots & & \end{bmatrix}$$

Upon integration of (7.50) by parts it is found that

$$L_{kn} = f_k(z) \frac{df_n(z)}{dz} \Bigg|_{-1}^{1} - \int_{-1}^{1} \frac{df_n}{dz} \frac{df_k}{dz} dz$$

The first term on the right-hand side is zero from (7.43). A second partial integration of the second term gives

$$L_{kn} = -(f_k(z) \frac{df_n(z)}{dz} \Bigg|_{-1}^{1} - \int_{-1}^{1} f_n \frac{d^2 f_k}{dz^2} dz) = L_{nk}$$

Thus $L$ is a symmetric matrix.

The set $f_k(z) = \cos[(2k - 1)\pi z]$ satisfies the requirements of (7.43) and (7.44). Since these functions are also the eigenfunctions for $\alpha = 0$, the matrix $L$ is diagonal with the elements given by

$$L_{kk} = -(2k - 1)^2 \pi^2/4$$

The elements of M are found to be

$$M_{kn} = (-1)^{k-1} \frac{(2k-1)(2n-1)}{(2k-1)^2 - (2n-1)^2} \qquad k > n$$

$$M_{nk} = -M_{kn}$$

$$M_{kk} = -\frac{1}{2}$$

Also $$c_k = 1$$

and $$q_k(0) = \frac{4}{\pi} \frac{(-1)^{k-1}}{2k-1}$$

The matrices and vectors of (7.47) are thus:

$$C = I$$

$$M = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & -\frac{5}{12} & +\frac{7}{24} & \cdots \\ -\frac{3}{4} & -\frac{1}{2} & +\frac{15}{8} & -\frac{21}{20} & \cdots \\ \frac{5}{12} & -\frac{15}{8} & -\frac{1}{2} & \frac{35}{12} & \cdots \\ -\frac{7}{24} & \frac{21}{20} & -\frac{35}{12} & -\frac{1}{2} & \cdots \\ & \cdots & & & \\ & \cdots & & & \end{bmatrix}$$

$$L = -\frac{\pi^2}{4} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 9 & 0 & 0 & \cdots \\ 0 & 0 & 25 & 0 & \cdots \\ 0 & 0 & 0 & 49 & \cdots \\ & \cdots & & & \\ & \cdots & & & \end{bmatrix}$$

$\underline{q}'(0) = \frac{4}{\pi}\left[1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7} \dots\right]$

Approximate solutions were computed for the cases tabulated below

N = 1, 2, 3, 4, 5 $\qquad\qquad\qquad\qquad$ $\alpha$ = 0.0, 0.5, 1.0, 2.0, 4.0

The results obtained, may be summarized as follows:

(a) For small values of time (or for probabilities greater than 0.8) convergence to the solution tends to be slow. This is mainly due to slow convergence of the cosine expansion of the constant initial condition.

(b) For the case $\alpha = 2aB^2/\sigma^2 = 0$ the method tends to give accurate results. The exact half life for this case is 0.375; using cosines we obtain 0.380.

(c) As the magnitude of $\alpha$ increases, the rate of convergence decreases. Consequently, for large values of $\alpha$, more terms are required.

Using the cosine expansion for N = 5 we have plotted the confinement probability for $\alpha$ = 0, 1, 2, 4 in Figure 7-5. Clearly the confinement probability increases with $\alpha$. The half-life, obtained from these curves, is plotted as a function of normalized system gain $\alpha$ in Figure 7-4. Note that the average first passage time and the exact half-life have approximately the same general behavior with respect to system gain, and hence that the mean passage time can be used to estimate the half-life.

The lower bound on the confinement probability (6.5) is obtained by using the principal hypergeometric eigenfunction for V(y) and the principal eigenvalue for $\alpha$. These values are tabulated in [10]. For $\alpha$ = 1, this "principal eigenvalue" lower bound is shown in Figure 7-6. The upper bound (6.1) is given by

$$\hat{q}(t|0) = \operatorname{erf}\left[\left(\frac{\sigma^2}{aB^2}\left[-1 + \exp(2at)\right]\right)^{-1/2}\right] \qquad (7.53)$$

and is also plotted in Figure 7-6.

**FIGURE 7-5**
**LINEAR SYSTEM CONFINEMENT PROBABILITY**

$\alpha = 2aB^2/\sigma^2, \quad \dot{X} = -aX + \sigma\xi$

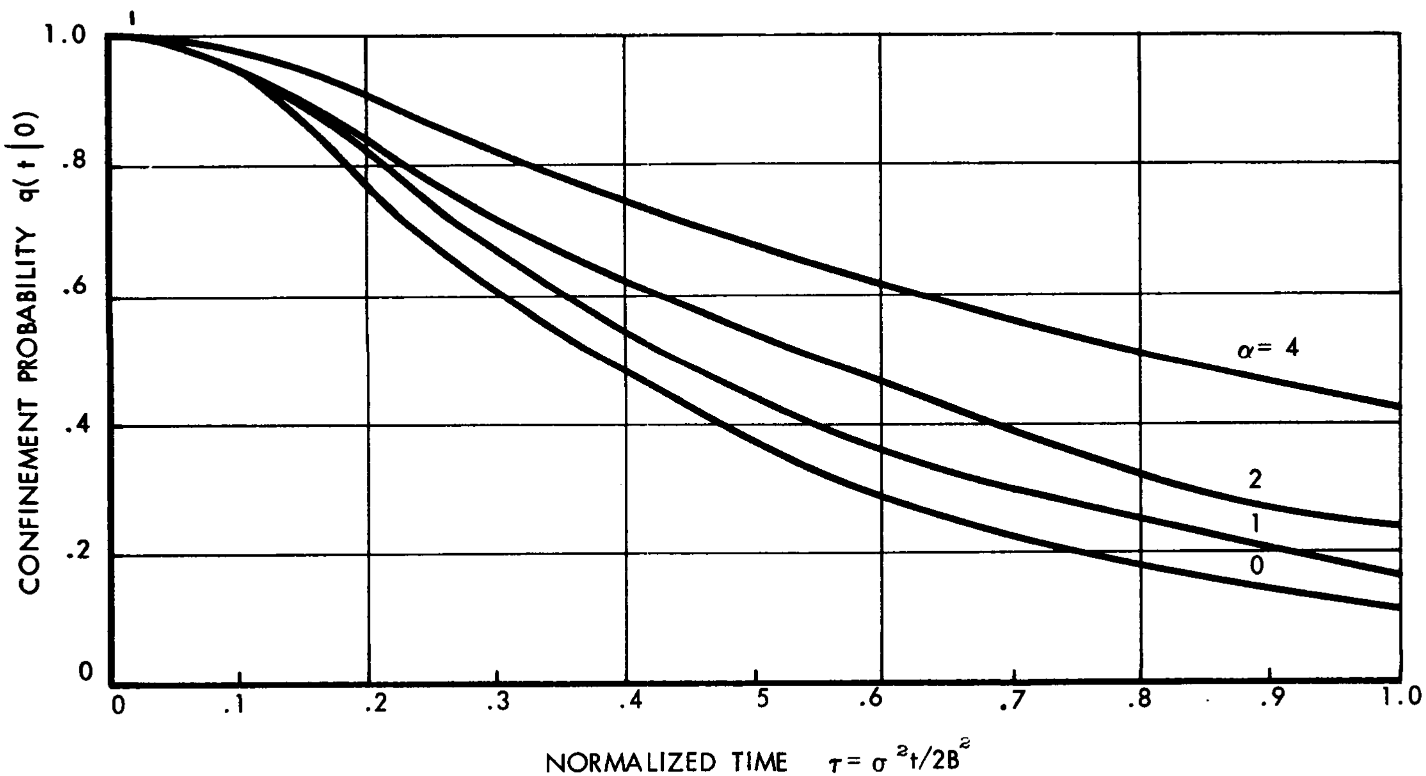
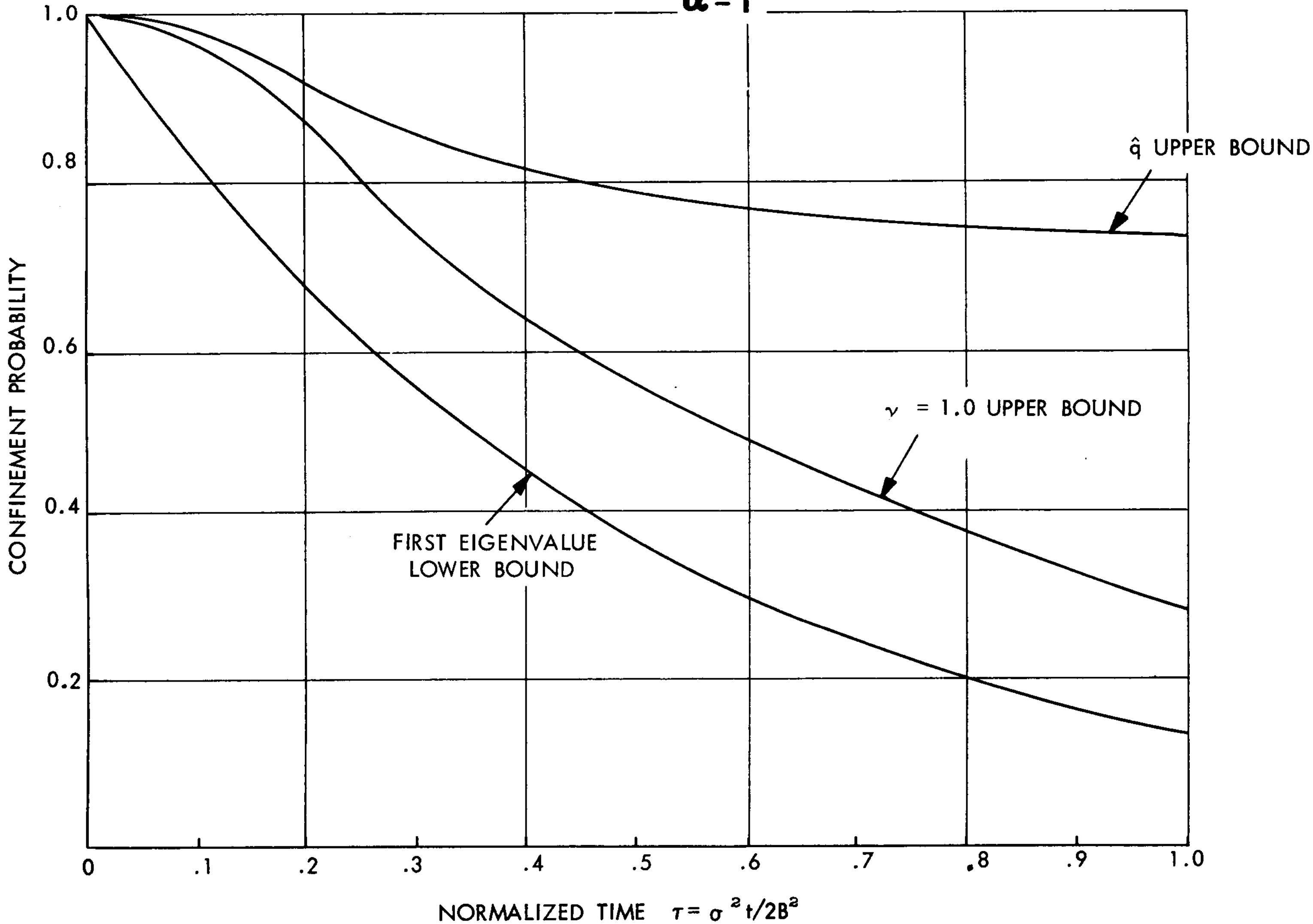


**FIGURE 7-6**

**ESTIMATES OF CONFINEMENT PROBABILITY FOR LINEAR SYSTEM**

**$\alpha = 1$**



An approximate solution to the first-order equation

$$\frac{\partial q}{\partial t} = -f(y)\,\frac{\partial q}{\partial y} + \frac{\sigma^2}{2}\,\frac{\partial^2 q}{\partial y^2} \tag{7.54}$$

can be obtained as follows: introduce the normalization.

$$\begin{aligned} \tau &= \sigma^2 t/B^2 \\ \eta &= y/B \end{aligned} \tag{7.55}$$

This reduces (7.54) to

$$\frac{\partial q}{\partial \tau} = -\hat{f}(\eta)\,\frac{\partial q}{\partial \eta} + \frac{1}{2}\,\frac{\partial^2 q}{\partial \eta^2} \tag{7.56}$$

where

$$\hat{f}(\eta) = \frac{B}{\sigma^2}\, f(B\eta) \tag{7.57}$$

with the boundary conditions

$$\begin{aligned} q(0|\eta) &= 1 \\ q(\tau|\pm 1) &= 0 \end{aligned} \tag{7.58}$$

We will now show that (7.56) can be reduced to a diffusion equation.

Let

$$z = \varphi(\eta)$$

and

$$g(\eta) = \frac{dz}{d\eta} = \varphi'(\eta)$$

Then

$$\frac{\partial q}{\partial \eta} = \frac{\partial q}{\partial z}\,\frac{\partial z}{\partial \eta} = g(\eta)\,\frac{\partial q}{\partial z}$$

amd

$$\frac{\partial^2 q}{\partial \eta^2} = g'(\eta)\,\frac{\partial q}{\partial z} + [g(\eta)]^2\,\frac{\partial^2 q}{\partial z^2}$$

Substitute the above two expressions into (7.56) to obtain

$$\frac{\partial q}{\partial \tau} = -\hat{f}(\eta)\, g(\eta) \frac{\partial q}{\partial z} + \frac{1}{2} [g'(\eta) \frac{\partial q}{\partial z} + [g(\eta)]^2 \frac{\partial^2 q}{\partial z^2}$$

$$= [-\hat{f}(\eta)\, g(\eta) + \frac{1}{2} g'(\eta)] \frac{\partial q}{\partial z} + \frac{1}{2} [g(\eta)]^2 \frac{\partial^2 q}{\partial z^2} \tag{7.59}$$

Now select $g(\eta)$ to satisfy the differential equation

$$g'(\eta) = 2\hat{f}(\eta)\, g(\eta)$$

and we have

$$g = K \exp [2 \int_0^\eta \hat{f}(w)\, dw] \tag{7.60}$$

Then (7.59) becomes the diffusion equation

$$\frac{\partial q}{\partial \tau} = \frac{1}{2} [\bar{g}(z)]^2 \frac{\partial^2 q}{\partial z^2} \tag{7.61}$$

where $\qquad \bar{g}(z) = g(\eta(z))$

subject to the boundary conditions

$$q(0|z) = 1$$

$$q(\tau|\pm Z) = 0 \tag{7.62}$$

where $\qquad Z = \varphi(1) = K \int_0^1 \exp [2 \int_0^\eta \hat{f}(u)\, du]\, d\eta \qquad (7.63)$

Introduce a further normalization:

$$\zeta = \frac{z}{Z}$$

Then (7.61) becomes

$$\frac{\partial q}{\partial \tau} = \frac{1}{2} \gamma^2(\zeta) \frac{\partial^2 q}{\partial \zeta^2} \tag{7.64}$$

where $$\gamma(\zeta) = \frac{\bar{g}(Z\zeta)}{Z} \tag{7.65}$$

subject to the boundary conditions

$$\begin{aligned} q(0|\zeta) &= 1 \\ q(\tau|\pm 1) &= 0 \end{aligned} \tag{7.66}$$

Note that if

$$\Gamma_\ell < \gamma(\zeta) < \Gamma_u \quad \text{for} \quad |\zeta| < 1$$

then the solution to (7.64) is bounded between $q_\ell$ and $q_u$, where $q_\ell$ is the solution to (7.64) with $\gamma$ replaced by $\Gamma_\ell$, and $q_u$ is the solution to (7.63) with $\gamma$ replaced by $\Gamma_u$.

With $\gamma = \Gamma = \text{const}$, the solution to (7.64) can be expressed in the form

$$q(\tau|\zeta) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left[-\frac{(2k+1)^2 \pi^2 \Gamma^2 \tau}{8}\right] \cos\left[(2k+1)\frac{\pi}{2}\zeta\right] \tag{7.67}$$

Computationally, this solution is best for large times, and

$$q(\tau|0) \approx \frac{4}{\pi} \exp\left(-\frac{\pi^2 \Gamma^2 \tau}{8}\right) \quad \text{for large } \tau \tag{7.68}$$

For small times, the following form is preferable

$$q(\tau|0) = 1 - 2 \sum_{n=0}^{\infty} (-1)^n \operatorname{erfc}\left[\frac{2n+1}{\Gamma\sqrt{2\tau}}\right] \tag{7.69}$$

$$\approx 1 - 2 \operatorname{erfc}\left[\frac{1}{\Gamma\sqrt{2\tau}}\right] \quad \text{for small } \tau \tag{7.70}$$

In a linear system

$$F(y) = ay \tag{7.71}$$

Hence $$f(\eta) = \frac{aB^2\eta}{\sigma^2}$$

and, if we let $\alpha = aB^2/\sigma^2$ we have

$$g(\eta) = K \exp\left[2 \int_0^{\eta} \alpha u \, du\right] = K \exp\left[\alpha \eta^2\right] \tag{7.72}$$

and

$$z = K \int_0^{\eta} e^{\alpha x^2} dx \tag{7.73}$$

It is convenient to define $K = \sqrt{\alpha}$, so

$$g = \sqrt{\alpha}\; e^{\alpha \eta^2} = \sqrt{\alpha}\; e^{u^2} \tag{7.74}$$

$$z = \int_0^{\eta} e^{\alpha \eta^2} d(\sqrt{\alpha}\eta) = \int_0^{u} e^{x^2} dx \tag{7.75}$$

where

$$u = \eta\sqrt{\alpha} \tag{7.76}$$

Then (7.64) becomes

$$\frac{\partial q}{\partial \tau} = \frac{1}{2} \gamma^2(\zeta) \frac{\partial^2 q}{\partial \zeta^2}$$

where

$$\gamma = \frac{\sqrt{\alpha}\zeta}{F(\sqrt{\alpha}\eta)} = \frac{\sqrt{\alpha}\; e^{\eta^2 \alpha}}{\int_0^{\sqrt{\alpha}} e^{x^2} dx} \tag{7.77}$$

and

$$\zeta = \frac{F(\sqrt{\alpha}\eta)}{e^{\alpha(1-\eta^2)} F(\sqrt{\alpha})} \tag{7.78}$$

An upper bound to the confinement probability for the linear system may be obtained by using $\gamma = 1.0$, thus bounding the solution from above by the solution to a diffusion process shown in Figure 7-6. From the figure it can be seen that the $\gamma = 1.0$ upper bound is a considerable improvement over the $\hat{q}(t|0)$ upper bound.

## 7.4 First-Order Bang-Bang System

Consider the system

$$\dot{x} = -c \operatorname{sgn} x + \xi \tag{7.79}$$

where $\xi$ is white noise with variance $\sigma^2\ \delta(t)$. The confinement probability satisfies

$$\frac{\partial q}{\partial t} = -\ c \operatorname{sgn} y \frac{\partial q}{\partial y} + \frac{\sigma^2}{2} \frac{\partial^2 q}{\partial y^2} \tag{7.80}$$

subject to conditions (7.3) and (7.4). Take the Laplace transform of (7.80) and apply initial condition (7.4) to obtain

$$\frac{d^2 \tilde{q}}{dy^2} - \frac{2c}{\sigma^2} \operatorname{sgn} y \frac{d\tilde{q}}{dy} - \frac{2s}{\sigma^2} \tilde{q} = -\frac{2}{\sigma^2} \tag{7.81}$$

Let $\tilde{q}_1(s|y)$ represent the solution is the interval $0 < y < B$. Then

$$\tilde{q}_1(s|y) = A_1 e^{\lambda_1 y} + A_2 e^{\lambda_2 y} + \frac{1}{s} \tag{7.82}$$

where

$$\lambda_1 = \frac{c}{\sigma^2} + \frac{1}{\sigma}(2s + c^2/\sigma^2)^{1/2} \ ; \ \lambda_2 = \frac{c}{\sigma^2} - \frac{1}{\sigma}(2s + c^2/\sigma^2).^{1/2}$$

Using the boundary condition $\tilde{q}_1(s/B) = 0$, we have

$$A_2 = -\left(\frac{1}{s} + A_1 e^{\lambda_1 B}\right) e^{-\lambda_2 B} \tag{7.83}$$

Hence (7.82) becomes

$$\tilde{q}_1(s|y) = \frac{1}{s} + A_1 e^{\lambda_1 y} - \left(\frac{1}{s} + A_1 e^{\lambda_1 B}\right) e^{\lambda_2 (y - B)} \ , \quad 0 < y < B \tag{7.84}$$

Now let $\tilde{q}_2(s|y)$ represent the solution to (7.80) in the interval $-B < y < 0$.

$$\tilde{q}_2(s|y) = A_3 e^{\lambda_3 y} + A_4 e^{\lambda_4 y} + \frac{1}{s} \tag{7.85}$$

where

$$\lambda_3 = -\frac{c}{\sigma^2} + \frac{1}{\sigma}(2s + c^2/\sigma^2)^{1/2} \text{ and } \lambda_4 = -\frac{c}{\sigma^2} - \frac{1}{\sigma}(2s + c^2/\sigma^2)^{1/2} .$$

Using the boundary condition $\tilde{q}_2(s|-B) = 0$, we have

$$A_4 = -\left(\frac{1}{s} + A_3 e^{-\lambda_3 B}\right) e^{+\lambda_4 B} \tag{7.86}$$

Equation (7.84) becomes

$$\tilde{q}_2(s|y) = \frac{1}{s} + A_3 e^{\lambda_3 y} - \left(\frac{1}{s} + A_3 e^{-\lambda_3 B}\right) e^{\lambda_4 (y+B)}, \quad -B < y < 0 \tag{7.87}$$

Since the required solution must be differentiable at the origin and symmetric, we must have

$$\left.\frac{d\tilde{q}_1(s|y)}{dy}\right|_{y=0} = \left.\frac{d\tilde{q}_2(s|y)}{dy}\right|_{y=0} = 0 \tag{7.88}$$

Therefore,

$$A_1 = \frac{1}{s} \frac{\lambda_2}{\lambda_1 e^{\lambda_2 B} - e^{\lambda_1 B} \lambda_2} \tag{7.89}$$

$$A_3 = \frac{1}{s} \frac{\lambda_4}{\lambda_3 e^{-\lambda_4 B} - \lambda_4 e^{-\lambda_3 B}} \tag{7.90}$$

and, after some manipulation, $\tilde{q}_1(s|y)$ and $\tilde{q}_2(s|y)$ become

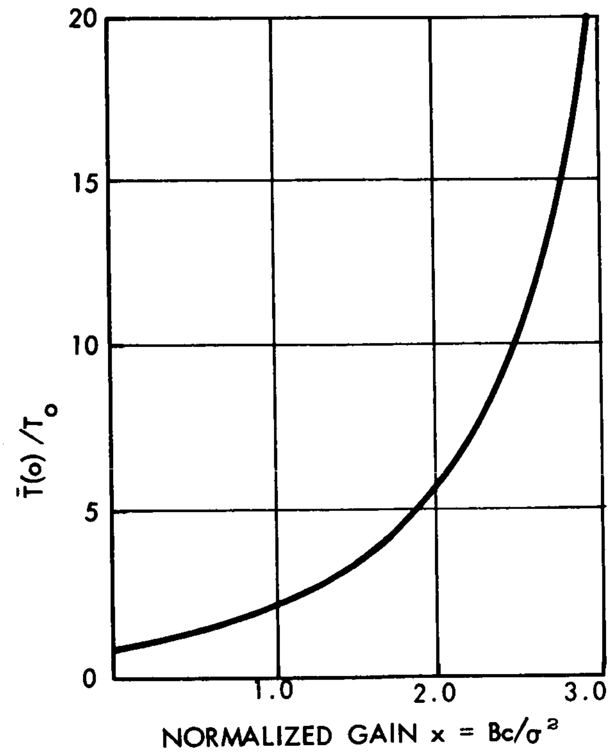
$$\tilde{q}_1(s|y) = \frac{1}{s}\left(1 - \frac{\lambda_2 e^{\lambda_1 y} - \lambda_1 e^{\lambda_2 y}}{\lambda_2 e^{\lambda_1 B} - \lambda_1 e^{\lambda_2 B}}\right), \quad 0 < y < B \tag{7.91}$$

$$\tilde{q}_2(s|y) = \frac{1}{s}\left(1 - \frac{\lambda_4 e^{\lambda_3 y} - \lambda_3 e^{\lambda_4 y}}{\lambda_4 e^{-\lambda_3 B} - \lambda_3 e^{-\lambda_4 B}}\right), \quad -B < y < 0 \tag{7.92}$$

Note that $\lambda_4 = -\lambda_1$ and $\lambda_3 = -\lambda_1$. Therefore, the above two equations may be combined to yield

**FIGURE 7-7**

**RATIO OF EXPECTED FIRST PASSAGE TIME FOR FIRST-ORDER BANG-BANG SYSTEM TO EXPECTED FIRST PASSAGE TIME FOR INTEGRATOR**



$$\mathcal{q}(s|y) = \frac{1}{s}\left(1 - \exp\left[-\alpha B\left(1 - \frac{|y|}{B}\right)\right]\frac{\cosh \beta y - \frac{\alpha}{\beta}\sinh \beta y}{\cosh \beta B - \frac{\alpha}{\beta}\sinh \beta B}\right), \quad -B < y < B \qquad (7.93)$$

where

$$\alpha = \frac{c}{\sigma^2}$$

$$\beta = \frac{c}{\sigma^2}\left(1 + \frac{2s\sigma^2}{c^2}\right)^{1/2}$$

It is noted that (7.92) is again a meromorphic function of $s$. Accordingly, following the procedure of example 7.1, the inverse Laplace transform of (7.92) gives

$$q(t|y) = \sum_{k=1}^{\infty} c_k e^{-s_k t} \exp\left[-\alpha B\left(1 - \frac{|y|}{B}\right)\right]\left(\cosh \beta_k y - \frac{\alpha}{\beta_k}\sinh \beta_k y\right) \qquad (7.94)$$

where the $s_k$ are the eigenvalues or poles of $\mathcal{q}(s|y)$, i.e., the zeros of the transcendental equation

$$\cosh \beta B - \frac{\alpha}{\beta}\sinh \beta B = 0$$

and $\beta_k$ are the corresponsing values of $\beta$.

The average first passage time can be found from (5.18) and (7.92) to be

$$\bar{T}(0) = T_0\left(\frac{e^{2x} - 2x - 1}{2x^2}\right) \qquad (7.95)$$

Where

$T_0 = \frac{B^2}{\sigma^2}$ = average first passage time for integrator (see (7.11)).

$$x = \frac{Bc}{\sigma^2}$$

The ratio $\bar{T}(0)/T_0$ is shown in Figure 7-7. Note that the average first passage time increases rapidly with $c$.

## 7.5 Double Integrator With White Noise Input

Consider the double integrator system

$$\begin{aligned} \dot{x}_1 &= \xi \\ \dot{x}_2 &= x_1 \end{aligned} \tag{7.96}$$

where $x_1$ and $x_2$ represent velocity and position respectively. Consider the problem of calculating the position confinement probability

$$q_2(t\,|y_1,y_2) = \text{Prob}\,\{\,|X_2(\lambda)| < B_2 \text{ for } \lambda \epsilon [0,t]\,|X_1(0) = y_1,\ X_2(0) = y_2\}$$

Note that $\{X_2\}$ is not a Markov process by itself; the process $\{X_1, X_2\}$, governed by (7.96) however is a two-dimensional Markov process, and hence $q_2$ satisfies the backward equation

$$\frac{\partial q_2}{\partial t} = y_1 \frac{\partial q_2}{\partial y_2} + \frac{\sigma^2}{2} \frac{\partial^2 q_2}{\partial y_1^2} \tag{7.97}$$

with the boundary conditions

$$q_2(t|y_1, \pm B_2) = 0, \quad \text{all } t,\ y_1 \tag{7.98}$$

$$\lim_{y_1 \to \pm\infty} q_2(t|y_1,y_2) = 0, \quad \text{all } t,\ y_2 \tag{7.99}$$

The problem (7.97)-(7.98) does not appear amenable to exact solution, and we thus proceed to compute bounds by the methods of Section 6. The upper bound (6.1) is

$$\hat{q}(t|0) = \text{Prob}\,\{|X_2(t)| < B_2 | X_2(0) = 0\}$$

This is easily calculated from

$$\hat{q}(t|0) = \int_{-\infty}^{\infty} \int_{-B_2}^{B_2} p(x_1,x_2,t|0)\, dx_2)\, dx_1 \tag{7.100}$$

where, by use of (4.13) and (4.15), it is found that

$$p(x_1,x_2,t|0) = \frac{1}{2\pi(\sigma^2 t^2/2\sqrt{3})} \exp\left[-\tfrac{1}{2} \frac{12}{\sigma^4 t^4}\left(\frac{\sigma^2 t^3}{3} x_1^2 - \sigma^2 t^2 x_1 x_2 + \sigma^2 t x_2^2\right)\right] \tag{7.101}$$

Substitute (7.101) into (7.100) to obtain

$$\hat{q}(t|0) = \text{erf}\ [\sqrt{3}/2\tau^{3/2}] \tag{7.102}$$

where $\tau$ is the normalized time, defined by

$$\tau = \left[\frac{\sigma^2}{2B_2^2}\right]^{1/3} t \tag{7.103}$$

We have not been successful in obtaining a lower bound by the methods of Wonham or Kushner. However, for this problem a lower bound can be found by the third approach described in Section 6. This method consists of seeking a solution in the form of the characteristic function (4.19). Specifically, we try a solution of the form

$$q_c(t|y) = \sum_{m,n} A_{mn} \exp\ [-\tfrac{1}{2}\nu'_{mn} M \nu_{mn} + j\nu'_{mn}\mu] \tag{7.104}$$

where

$$M = \begin{bmatrix} \sigma^2 t & \sigma^2 t^2/2 \\ \sigma^2 t^2/2 & \sigma^2 t^3/3 \end{bmatrix}$$

is the solution of the variance equation (4.15) for the problem and

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_1 t + y_2 \end{bmatrix}$$

is the conditional mean, satisfying (4.14), and $\nu'_{mn} = [\ell_n, k_m]$ with $\ell_n$ and $k_m$ being arbitrary constants. Since each term of (7.104) is a characteristic function for the conditional transition density function, it follows that $q_c$ satisfies the backward equation satistied by $q_2$. Thus, if the boundary conditions (7.98) and (7.99) for $q_2$ could be satisfied by $q_c$ by an appropriate choice of $\ell_n$ and $k_m$, the problem would be solved. Because $\mu_2$ is a function of time, however, it is not possible to satisfy (7.98) and (7.99) by $q_c$. Rather, it is possible to satisfy the following condition:

$$q_c(t|\pm B_1, y_2) = q_c(t|y_1, \pm(B_2 - |y_1|t)) = 0 \tag{7.105}$$

It is clear that the region enclosed by the boundaries defined by (7.105) is contained within the boundaries defined by (7.98) and (7.99) and hence $q_c$ serves as a lower bound on $q_2$. Upon expression of the complex exponentials in terms of sines and cosines, and after using the boundary conditions (7.105) and the initial condition

$$q_c(0|y_1,y_2) = \begin{cases} 1 & |y_1| < B_1\,, \quad |y_2| < B_2 \\ 0 & \text{elsewhere} \end{cases}$$

it is found that

$$q_c(t|y_1,y_2) = \frac{16}{\pi^2} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^{k+\ell}}{(2k+1)(2\ell+1)} e^{-\frac{\pi^2\sigma^2}{8}\alpha_{k\ell}(t)} \cos\left(\frac{2k+1}{2B_1}\pi y_1\right)$$

$$\cdot \cos\left[\frac{2\ell+1}{2B_2}\pi\,(y_2 + y_1 t)\right] \tag{7.106}$$

where

$$\alpha_{k\ell}(t) = \frac{(2k+1)^2}{B_1^2}t + \frac{(2k+1)(2\ell+1)}{B_1B_2}t^2 + \frac{(2\ell+1)^2}{3B_2^2}t^3$$

(Note that as $B_2 \to \infty$, $q_c \to q$ of (7.8) as it should.) Now, from (7.106) as $B_1 \to \infty$ the lower bound for $q_2(\tau|0)$ becomes

$$q_c(\tau|0) = \frac{4}{\pi} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{2\ell+1} \exp\left[-\frac{\pi^2\tau^3}{12}(2\ell+1)\right]$$

with $\tau$ given by (7.103). This lower bound, along with the upper bound (7.102) and the results of a Monte-Carlo simulation are shown in Figure 7-8. The figure indicates that the normalized half-life, with respect to the position limit $B_2$, is $\tau_h = 1$, or

$$t_h = \left[\frac{2B_2^2}{\sigma^2}\right]^{1/3} \tag{7.107}$$

Galerkin's method can be applied to this problem by replacing the boundary condition (7.99) by

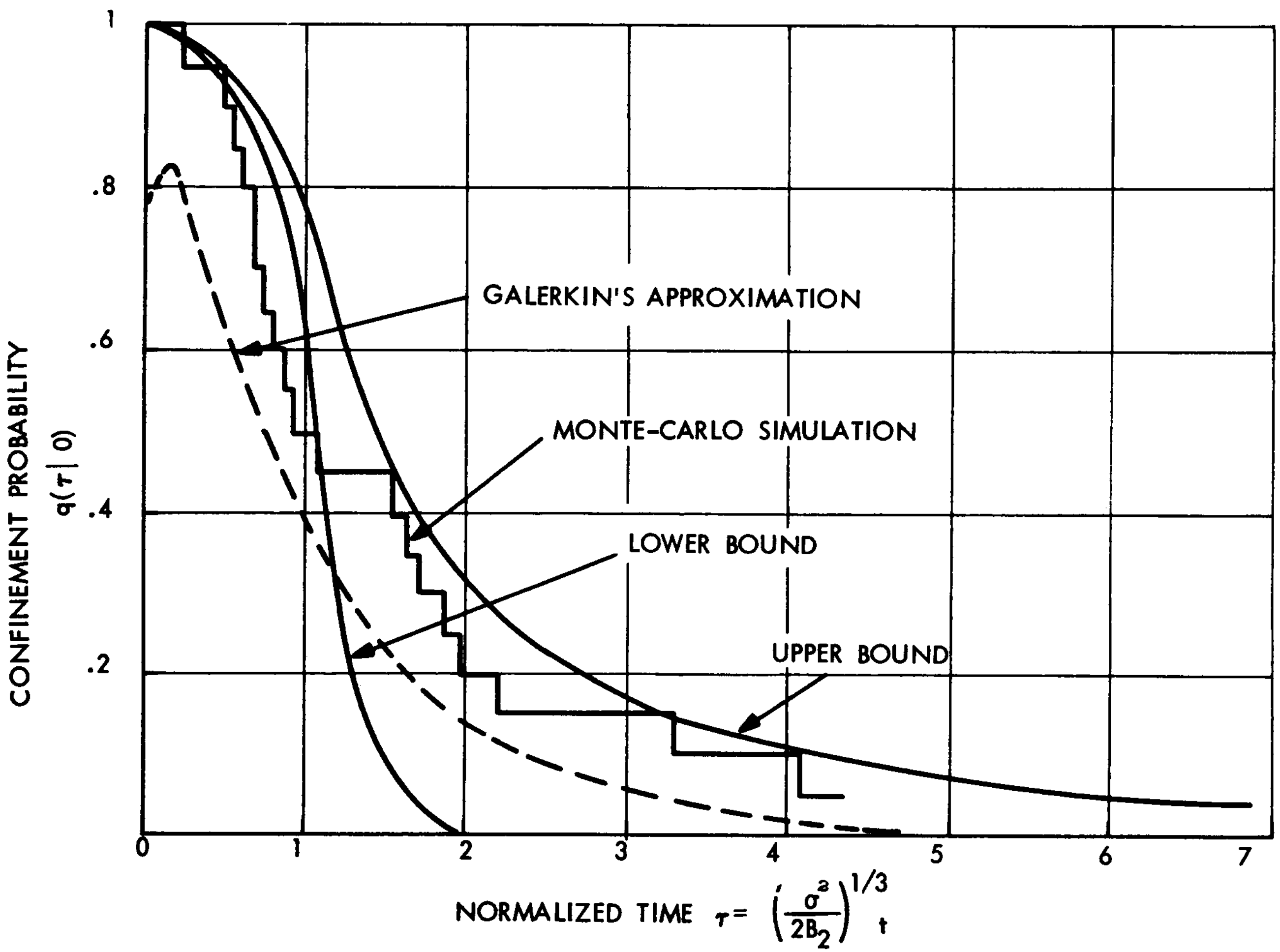
$$q(t|\pm B_1,y_2) = 0 \qquad \text{with} \quad B_1 >> 1$$

and using the initial condition

$$q(0|y_1,y_2) = 1 \qquad \text{for } |y_1| < B_1, \ |y_2| < B_2\,.$$

This represents a lower bound on $q_2(t|y_1,y_2)$.

**FIGURE 7-8**
**DOUBLE-INTEGRATOR CONFINEMENT PROBABILITY**
$\ddot{x} = \sigma \xi$



Using sines and cosines as the orthogonal functions in each direction we assume a solution of the form

$$q(t|y_1,y_2) = \sum_{\ell j} \left[\alpha_{\ell j}(t) \sin \frac{\ell \pi x}{B_1} \sin \frac{j \pi y}{B_2} + \beta_{\ell j}(t) \cos \frac{(2\ell - 1)\pi x}{2B_1} \cos \frac{(2j - 1)\pi y}{2B_2}\right]$$

The ordinary differential equations for the time functions $\alpha_{\ell j}(t)$ and $\beta_{\ell j}(t)$ can be expressed in the following form:

$$\dot{\alpha}_{\ell j} = k_\ell \alpha_{\ell j} + \sum_{k,\ell} C^{\ell j}_{\ell k} \beta_{\ell k}$$

$$\dot{\beta}_{\ell j} = \sum_{k,\ell} C^{k\ell}_{j\ell} \alpha_{\ell k} + h_\ell \beta_{\ell j}$$

where

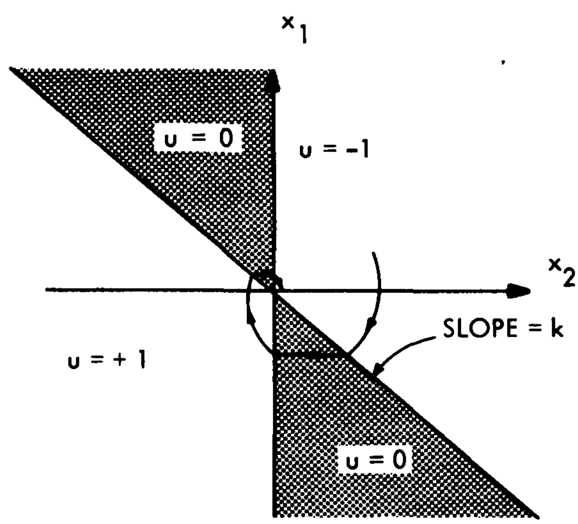
$$k_\ell = -\frac{\sigma^2 \ell^2 \pi^2}{2B_1^2}$$

$$h_\ell = -\frac{\sigma^2 (2\ell - 1)^2 \pi^2}{8B_1^2}$$

$$C^{\ell j}_{\ell k} = (-1)^\nu \frac{128}{\pi^2} \frac{B_1}{B_2} \frac{\ell j (2k - 1)(2\ell - 1)}{[(2k - 1)^2 - 4j^2][4\ell^2 - (2\ell - 1)^2]^2}$$

in which $\nu$ is an integer which depends on $\ell$, $j$, k, and $\ell$. A numerical solution shown in Figure 7-8 was computed for $N = 2$ with $|B_1| = |B_2| = 1$. This result appears to lie between the upper and lower bounds for values of normalized time greater than 1.2.

**FIGURE 7-9**
**NONLINEAR CONTROL SYSTEM**



## 7.6 Second-Order Nonlinear System

Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= Au(x_1,x_2) + \xi \\ \dot{x}_2 &= x_1 \end{aligned} \tag{7.108}$$

where $x_1$ and $x_2$ represent velocity and position, respectively, $A$ is a constant, $\xi$ is a white noise random disturbance with variance $\sigma^2\delta(t)$, and

$$u(x_1,x_2) = \begin{cases} +1 & x_1 + kx_2 \le 0\,, \; x_2 < 0 \\ 0 & \text{for all other values of } x_1,\, x_2 \\ -1 & x_1 + kx_2 > 0\,, \; x_2 > 0 \end{cases} \tag{7.109}$$

is shown in Figure 7-9 along with a typical trajectory in the deterministic case $(\xi \equiv 0)$.

Consider the problem of maintaining the position within bounds $|x_2| < B$. No analytic solution for the confinement probability has been obtained. Our attempts at obtaining estimates are illustrative of the difficulties involved in establishing quantitative measures of stochastic stability for nonlinear systems.

A Lyapunov function that was investigated is the total fuel consumption in the deterministic case,

$$V = k(x_1^2 + |x_2| + |x_2 + \tfrac{1}{k}x_1|) \tag{7.110}$$

It can be easily shown that in regions of constant torque, $\dot{V} = -1$, whereas in the coasting regions $\dot{V} = 0$. However, because of the corners which appear in curves of constant $V$, this function does not have the derivatives required by the theorems of Wonham and Kushner and thus cannot be used to obtain lower bounds on the confinement probability.

Monte-Carlo simulations of confinement probability vs. time for $A = 5 \times 10^{-4}$, $1 \times 10^{-4}$, $5 \times 10^{-5}$, $25 \times 10^{-6}$, and $5 \times 10^{-6}$ are shown in Figure 7-10. As expected the lifetime increases with the magnitude of the torque. A plot of the half-life vs. A for a fixed slope of 0.071 is shown in Figure 7-11. We have also obtained a plot of half-life as a function of $A$ for four values of the slope of the switching line. (See Figure 7-12.) In the region of acceleration magnitude between $1 \times 10^{-5}$ and $4.6 \times 10^{-5}$ the half-life varies inversely with the slope and increases with the

FIGURE 7-10
CONFINEMENT PROBABILITY

$$\ddot{\theta} + Au(\theta, \dot{\theta}) = \sigma\xi$$

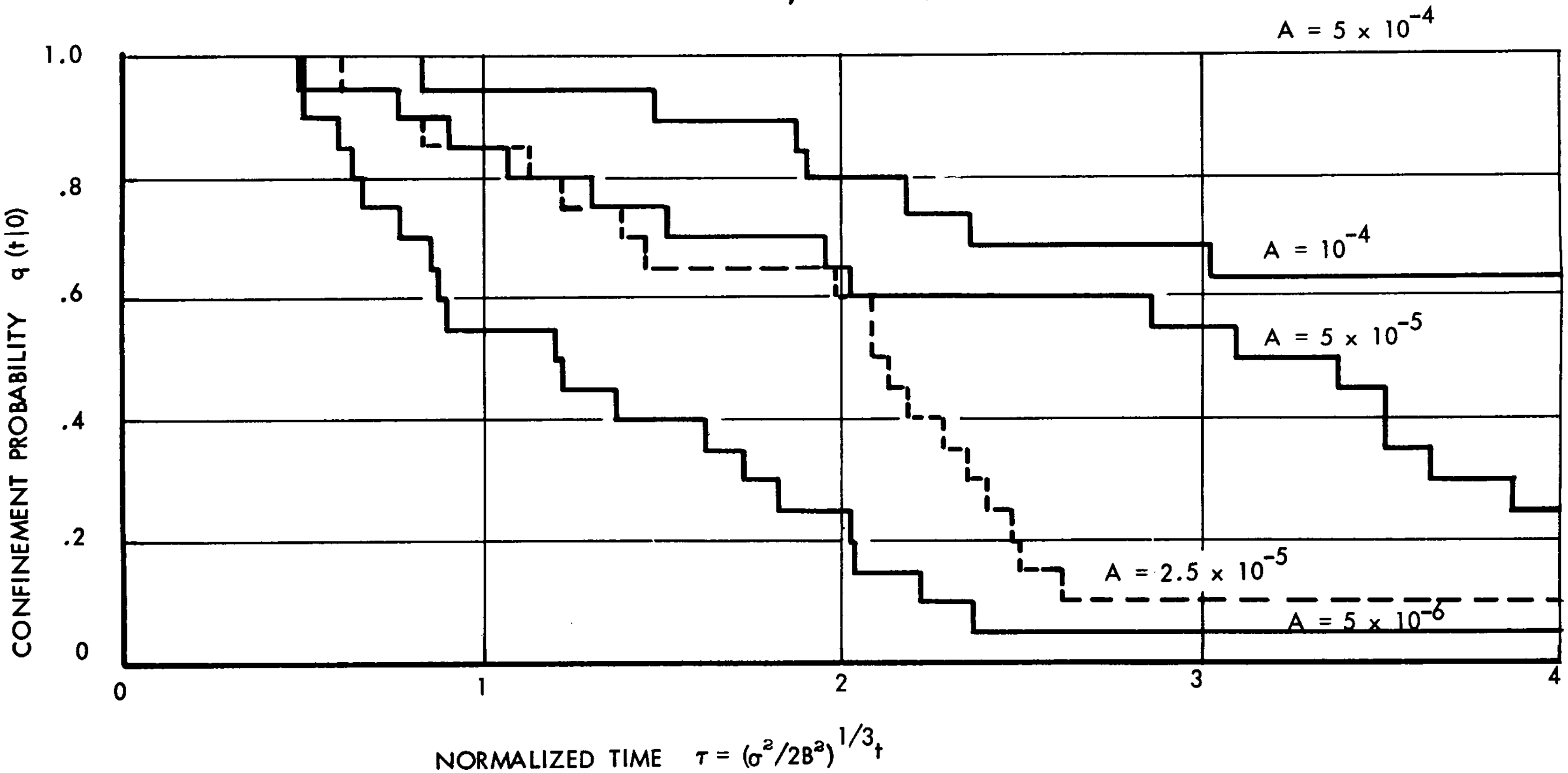
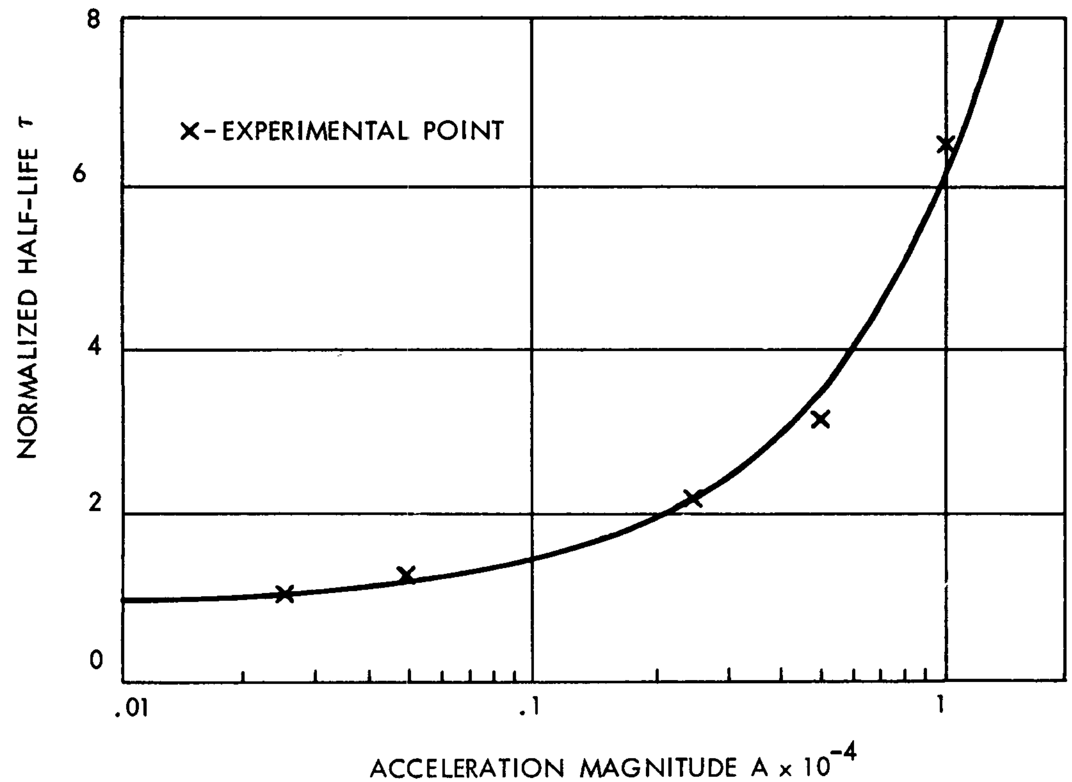
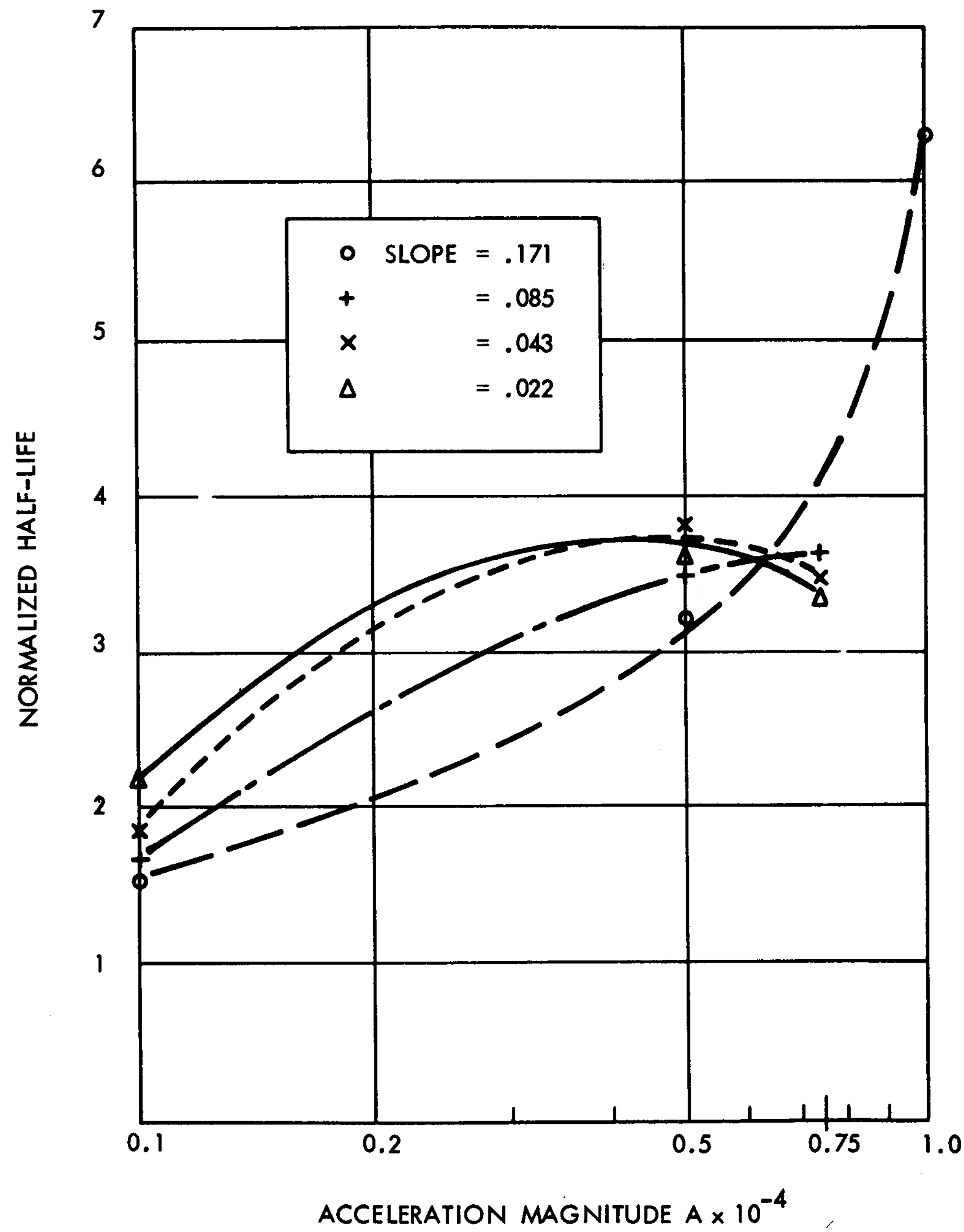


FIGURE 7-11
HALF-LIFE OF NONLINEAR CONTROL SYSTEM



## FIGURE 7-12
## HALF-LIFE OF NONLINEAR CONTROL SYSTEM



acceleration magnitude. At approximately $7 \times 10^{-5}$ the four curves merge indicating that at this value of acceleration the half-life is independent of slope. For values of slope of .02 and .04 the half-life curves reach maxima at $4.6 \times 10^{-5}$ and $5 \times 10^{-5}$, respectively, indicating that, at least for small values of slope, there is an "optimum" choice of acceleration magnitude corresponding to a given switching line slope.

Since it is relatively easy to obtain bounds on the confinement probability for linear systems, we have calculated upper bound (6.1) for the linear system

$$\begin{aligned} \dot{z}_1 &= \omega z_2 \\ \dot{z}_2 &= -\omega z_1 - 2\xi\omega z_2 + \frac{\sigma}{\omega}\,\xi \end{aligned} \tag{7.111}$$

where

$$z_1 \approx x_2 \qquad\qquad z_2 \approx \frac{x_1}{\omega} \tag{7.112}$$

The behavior of (7.111) is qualitatively similar to the nonlinear system (7.108); the constants $\xi$ and $\omega$ were determined in terms of the parameters of the nonlinear system as follows: Assume that with $\xi \equiv 0$ the nonlinear system is initially in state $(x_{10}, 0)$ then assume

a) $z_1$ and $x_2$ attain their first maximum at the same time,

b) the maximum amplitude attained by $x_2$ is some multiple $\ell$ of the bound on the size of the region, and

c) $z_1$ and $x_2$ have the same change of magnitude in the first cycle.

These conditions lead to the following formulas

$$\xi^2 = \frac{L(k,A,B)}{4\pi^2 - L(k,A,B)} \tag{7.113}$$

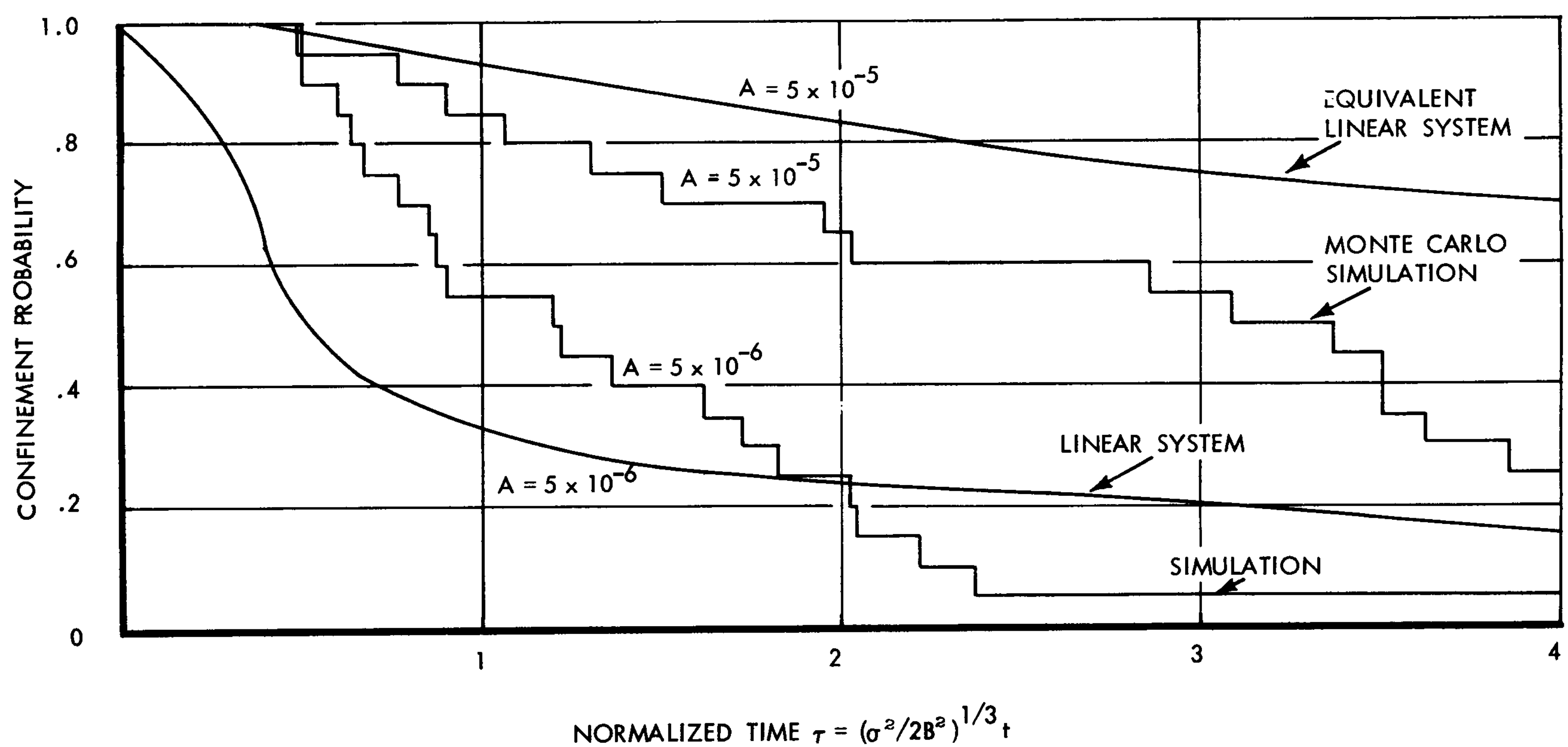
$$\omega = \frac{\pi}{\left[\frac{2B\ell}{A}\left(1 - \frac{L(k,A,B)}{4\pi^2 - L(k,A,B)}\right)\right]^{1/2}} \tag{7.114}$$

where

$$L(k,A,B) = \ell n \frac{\frac{2B\ell}{A}}{\left[-\frac{1}{k} + \left(\frac{3}{k^2} + \frac{2B\ell}{A} - \frac{2}{k}\left[\frac{1}{k^2} + \frac{2B\ell}{A}\right]^{1/2}\right)^{1/2}\right]^2} \tag{7.115}$$

**FIGURE 7-13**

**CONFINEMENT PROBABILITY OF NONLINEAR SYSTEM**



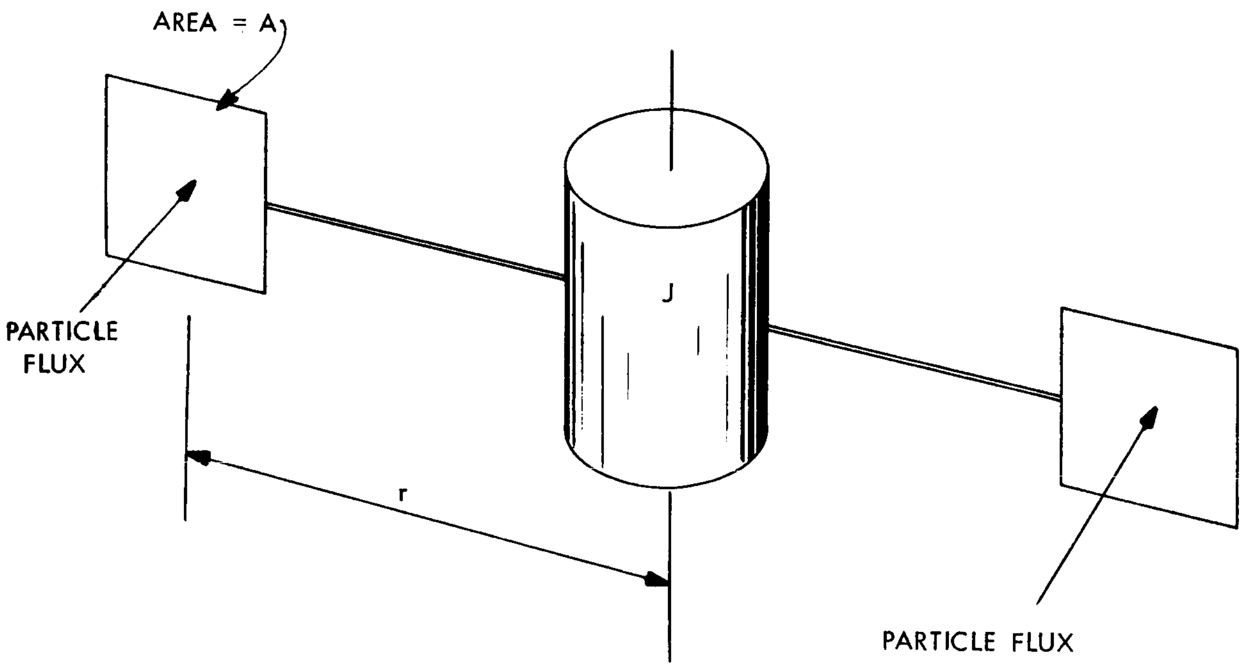
For the parameter values $k = .171$, $\frac{2B^2}{\sigma^2} = 200$, $\ell = 10$ we find that for $A = 5 \times 10^{-5}$ we have $\xi = .046$, $\omega = 3 \times 10^{-3}$ and for $A = \times 10^{-6}$ we have $\xi = .05$, $\omega = 3.83 \times 10^{-4}$. Using the same procedure as in (7.100)-(7.102) we find that $\hat{q}(t|0)$ , an upper bound to the confinement probability for the linear system, is as shown in Figure 7-13 along with the corresponding Monte-Carlo simulations. From the figure it can be seen that $\hat{q}(t|0)$ for the linear system has the same general behavior as the Monte-Carlo simulation and thus could possibly be used in estimating the lifetime of the non-linear system.

## 8. APPLICATIONS

One of the major practical problems in the application of the results of stochastic control theory is the establishment of the qualitative and quantitative characteristics of the sources of random excitation. As an example of how this problem might be approached, consider controlling the attitude of a satellite or space vehicle. The objective of the control system is to maintain a precise attitude (with respect to an appropriate reference system) in space, in the presence of random torques acting upon the vehicle. It is convenient to separate these random torques into two categories: torques due to the external environment acting on an "ideal" vehicle, and those due to the departure from the ideal. The torques due to the external environment are primarily the result of the bombardment of the vehicle by energetic particles of various types (photons, electrons, micrometeoroids, etc.) Torques due to departure from an ideal configuration might be caused by the leakage of gas from valves or holes in the vehicle, undesired motion of mechanical parts within the vehicle, or lack of symmetry with respect to the environment. The effect of a lack of symmetry is exemplified by the familiar radiometer, in which radiation pressure causes rotation of the unsymmetrically-reflective rotor structure.

In principle, it is possible to build a vehicle in which the torques due to the departure from the ideal are negligible in comparison to those due to the external environment, and it is thus of interest to consider the effect of the environmental torques on the motion of a typical vehicle. (In practice, the environmentally-caused torques may be many times smaller than those due to departure from the ideal, but the latter, which depend on many variable factors such as quality of construction, are difficult to assess.)

**FIGURE 8-1**
**IDEAL SPACE VEHICLE CONFIGURATION**



Consider an idealized space vehicle having the configuration shown in Fig. 8-1. Motion about one axis only will be considered here. It is assumed that the vehicle consists of a heavy core (with inertial J about the axis of rotation) of negligible area in comparison with two massless paddles, each of area A, rigidly fastened to the body by rods of length r. The random torques are assumed to be due to the random impact of many particles of various types; the average flux of which is uniform throughout space. (Solar radiation pressure is assumed to be the macroscopic effect of the impact of photons, which are assumed incident normal to the plane of the paddles. The reflectivities of the paddles are assumed to be identical.)

Each particle source is assumed to be statistically independent of all the others.

As a consequence of the assumption that the vehicle is dynamically balanced and that the particle flux is uniform, there is no mean torque on the vehicle, and hence no mean angular acceleration. In an infinitesimal time interval $\Delta t$, however, the actual particles impinging on the right hand paddle will invariably differ from those on the left and thus the vehicle will exhibit random rotation about the axis, and may thus wander from its equilibrium position. The first problem is thus to obtain a statistical model for the torques which cause this rotational vibration.

For deterministic torques acting upon the vehicle, the angular velocity $\omega$ about the axis of rotation is governed by the differential equation

$$J \; d\omega/dt = T(\omega, \theta, \ldots) + \ell \tag{8.1}$$

where J is the moment of inertia about the axis of rotation, T is the restoring torque developed by the vehicle control system, and $\ell$ is the total disturbance torque.

In the present case, however, the external torques are due to the superimposed random collisions of particles with the paddles. At the instant a particle strikes a paddle an impulsive torque is produced; at all other times the external torque is zero. It is thus more appropriate to consider the incremental change $d\omega$ in angular velocity, given by

$$J \; d\omega = T(\omega, \theta, \ldots) \, dt + d\pi \tag{8.2}$$

where $d\pi$ is the incremental angular momentum due to the random impact of particles in the differential time interval dt. For simplicity, we assume that the dimensions of the paddles are small compared with their distance from the axis; consequently all particles can be assumed to be acting at the center of the paddle. We thus can write

$$d\pi = r\ dw$$

where $dw$ is the total normal component of linear momentum transferred to the paddles in the interval $[t, t + dt]$. Thus (8.2) becomes

$$d\omega = \frac{T}{J}\ dt + \frac{r}{J}\ dw \tag{8.3}$$

and our problem is now to characterize the random process $dw$. On the basis of rather meager experimental evidence it appears reasonable to assume a Poisson distribution for the probability $P_n$ of the arrival of exactly in particles. Thus the particle bombardment can be modeled by a Poisson impulse process and the normal component of linear momentum can be characterized by the Poisson step process (2.4). The probability $\alpha(z)dz$ of (2.5) in the present situation denotes the probability that the normal component of linear momentum transferred by a single particle is between $z$ and $z + dz$. This probability depends on the nature of the surface, and may range from a fraction of the momentum of the particle to twice the particle momentum if the collisions are perfectly elastic.

If it is assumed that the momenta transferred by the individual particles are independent, the analysis in Section 2 is applicable and as $\Omega \to \infty$, where $\Omega$ is the average number of arrivals of particles in the interval $dt$, the probability density function $p(x, dt|y)$ is given by (2.10).

It is noted that $\sigma^2 = \lambda \hat{\sigma}^2$ (which, in this situation, has the dimensions of $(\text{momentum})^2/\text{time}$) is nonzero even though the vehicle is perfectly balanced, because of the random nature of the excitations. The representation of $dw$ as a Wiener process, however, permits the representation of the process by two parameters: the average arrival rate $\lambda$, and the mean-square linear momentum $\hat{\sigma}^2$. Because the exact nature of the probability distribution of momentum is generally not known, the evaluation of $\hat{\sigma}^2$ is a practical problem. In examining experimental data it may be easier to consider $\hat{\gamma}(x)$, the density function of a process causing rotation in only one direction, say clockwise. Let $\hat{Q}(\nu)$ be the characteristic function corresponding to $\hat{\gamma}(x)$. Assume that particles causing counterclockwise rotation have identical statistical properties as those causing clockwise rotation except that the probability density function for the counterclockwise linear momentum is $\hat{\gamma}(-x)$. Then the characteristic function for the counterclockwise process is $\hat{Q}(-\nu)$ and $Q(\nu) = \hat{Q}(\nu)\,\hat{Q}(-\nu)$. We find from (2.9) that

$$\hat{\sigma}^2 = 2\left[\int_0^\infty x^2 \hat{\gamma}(x)\,dx + \overline{m}^2\right] \geq 4\overline{m}^2$$

where

$$\bar{m} = \int_0^\infty x \, \hat{y}(x) \, dx$$

Therefore, the minimum value of $\sigma^2$ is $4\lambda \bar{m}^2$. Table 1 gives estimates of this value of $\sigma^2$ per unit area (A) for several sources of exciting particles.

TABLE 8-1

Estimates of Variance for Various Environmental Random Excitation Sources

| Source | $\varphi = \lambda/A$ particles/ sec-cm$^2$ | $\bar{m}$ dyne-sec | $\sigma^2/A$ (dyne-sec)$^2$/ sec-cm$^2$ |
|---|---|---|---|
| Photons | $2 \times 10^{18}$ | $2.5 \times 10^{-22}$ | $52 \times 10^{-26}$ |
| Nucleons | $1.5 \times 10^{7}$ | $2.2 \times 10^{-14}$ | $29 \times 10^{-21}$ |
| Micro-meteoroids | $10^{-7}$ | (See discussion below) | $22 \times 10^{-7}$ |

The calculation for photon was based on the assumption that $\bar{m} = 2 \times$ Planck's constant/wavelength with the average wavelength taken as $0.54 \times 10^{-4}$ cm; $\varphi$ was computed by assuming $\varphi = P/\bar{m}$ where P is the solar radiation pressure of 0.43 dynes/cm$^2$.

The computation for nucleons was made on the basis of the data in Reference 11 which corresponds to a period of fairly high solar activity.

The calculation for micrometeoroids was based on the following empirical law for the micrometeoroid flux inferred from Pegasus satellite measurements [26].

$$\psi = K\mu^{\beta} \qquad \beta < 1 \tag{8.4}$$

where $\psi$ is the flux (particles/$m^2$ -sec) of particles of mass $\geq \mu$, K and $\beta$ are constants which depend on the altitude of the vehicle. The upper and lower limits of mass are, respectively $\mu_1 = 1$ gm and $\mu_2 = 10^{-10}$ gm. Thus the number of particles having mass between $\mu$ and $\mu + d\mu$ is

$$d\psi = -K(\mu + d\mu)^{\beta} + K\mu^{\beta} = -K\beta\mu^{\beta-1}\, d\mu \tag{8.5}$$

The total flux $\varphi$ is obtained by integrating (8.5) between $\mu_1$ and $\mu_2$:

$$\varphi = \int_{\mu_1}^{\mu_2} d\psi = K(\mu_1^{\beta} - \mu_2^{\beta}) \tag{8.6}$$

For an altitude of 1000 km, $K \cong 10^{-15}$ and $\beta \cong -1.2$. Substitution of these values into (8.6) gives $\varphi = 10^{-3}$ particles/$m^2$ - sec = $10^{-7}$ particles/$cm^2$ - sec. To obtain the momentum probability distribution we assume that each particle has a constant velocity $V(\approx 27 \times 10^5$ cm/sec, Reference [26]). Then $d\psi$ of (8.5) is also the flux of particles having momentum between $\mu V$ and $\mu V + d(\mu V)$. The ratio of $d\psi$ to $\varphi$ is the probability density function for momentum. Hence

$$\int \hat{\gamma}(m)\, dm = \int \frac{\beta m^{\beta-1}\, dm}{m_2^{\beta} - m_1^{\beta}} \qquad m_1 \leq m \leq m_2 \tag{8.7}$$

$$m_1 = \mu_1 V$$
$$m_2 = \mu_2 V$$

The mean momentum is thus given by

$$\bar{m} = \int_{m_1}^{m_2} \frac{\beta m^{\beta}\, dm}{m_2^{\beta} - m_1^{\beta}} = \frac{\beta}{\beta + 1} \; \frac{m_2^{\beta+1} - m_1^{\beta+1}}{m_2^{\beta} - m_1^{\beta}}$$

and the variance is given by

$$\hat{\sigma}^2 = \int_{m_1}^{m_2} \frac{\beta m^{\beta+1} dm}{m_2^{\beta} - m_1^{\beta}} - \bar{m}^2$$

Using $m_1 = 27 \times 10^{-5}$ dyne-sec and $m_2 = 27 \times 10^{5}$ dyne-sec, and $\beta = -1.2$, we find that

$$\bar{m} = 1.6 \times 10^{-3} \text{ dyne-sec}, \ \hat{\sigma}^2 = 22 \text{ (dyne-sec)}^2$$

Because of the shape of the probability density function (8.7) $\bar{m}^2$ is much smaller than $\hat{\sigma}^2$. (The same problem could very well arise for other sources of excitation when $\hat{\gamma}(m)$ is highly skewed.)

The confinement probability with respect to angular velocity for a spacecraft in the absence of any restoring torque can be calculated with the aid of (7.8) obtained in example 7.1. The velocity half-life is obtained from (7.19):

$$t_h = 0.75 \frac{B_1^2}{c^2} \tag{8.8}$$

where

$$c = \frac{r\sigma}{J}$$

$B_1$ = angular velocity limit

It is of interest to estimate this half life for a typical spacecraft such as Mariner IV, having the following dimensions

| | |
|---|---|
| area | $A = 8 \times 10^3 \text{ cm}^2$ |
| moment arm | $r = 200$ cm |
| moment of inertia | $J = 8 \times 10^8$ dyne-cm-sec. |

Using an angular velocity limit $B_1 = 2$ deg/sec. and the $\sigma^2/A$ ratio of Table 8.1, the angular velocity half-life, as a result of each environmental disturbance source of Table 8.1, is given in Table 8.2.

TABLE 8.2

2.0° / SEC HALF-LIFE OF ROTATIONAL INERTIA

| Partial Source | Half-Life |
|---|---|
| Photons | $10^{23}$ years |
| Nucleons | $10^{18}$ years |
| Micrometeoroids | $10^{4}$ years |

Evidently, these sources of excitation are not significant contributors to random wandering of angular velocity from the equilibrium condition.

With regard to angular position, the situation is quite different. From (7.107) in example 7.5 the angular position half life is given, approximately by

$$t_h = \left[ \frac{2B_2^2}{c^2} \right]^{1/3}$$

where $B_2$ is the angular position limit. For $B_2 = 1$ deg and the vehicle with the physical parameters listed above, the angular position half life computation yields the numbers listed in Table 8.3.

TABLE 8.3

1.0° HALF-LIFE OF ROTATIONAL INERTIA

| Particle Source | Half-Life |
|---|---|
| Photons | 400 years |
| Nucleons | 10 years |
| Micrometeoroids | 2 hours |

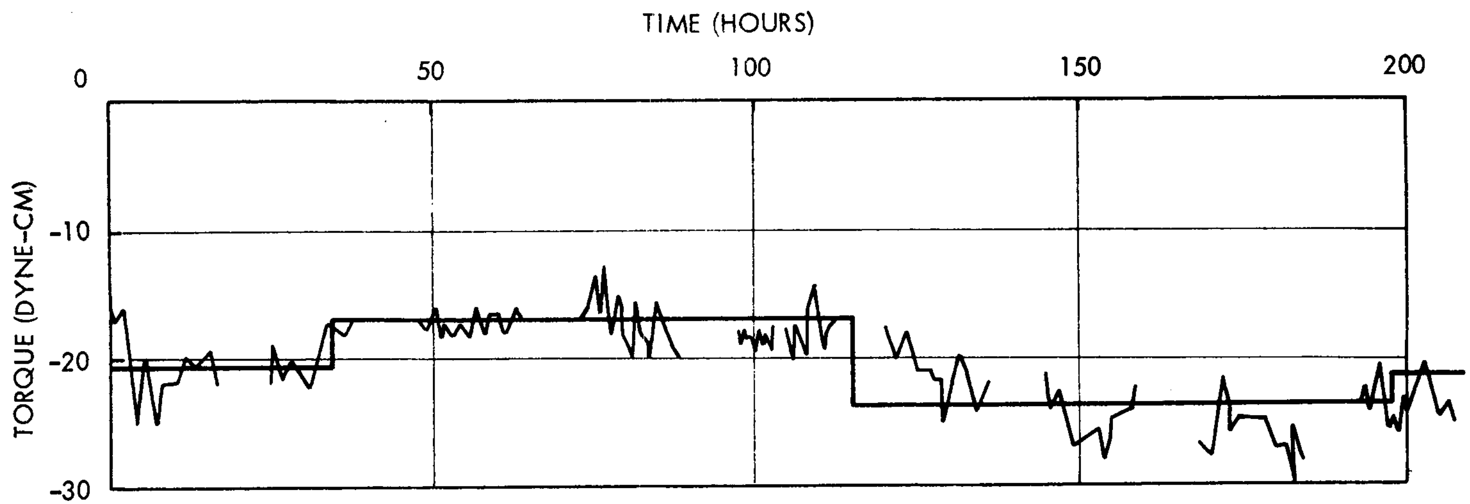
It is seen from Table 3 that micrometeoroids are likely to be a source of environmental disturbance of major practical concern. Photons and nucleons are not significant by comparison. It should be noted that the Wiener process which was used as a model for the source of random increments in angular momentum may be less than adequate for micrometeoroid bombardment; in the time interval equal to the computed half-life $t_h$ of 2 hours of Table 8.3 (based the Wiener process model), the expected number of particles is

$$N = \varphi A t_h \approx 6 \text{ particles}$$

where $\varphi$ is the micrometeoriod flux (Table 8.1). This is far from being a large number. In Example 7.2 it was determined that the half-life for angular velocity is conservatively estimated by using the Wiener process model, i.e., the actual half life is somewhat larger than computed using the Wiener process model; it is not presently known, however, whether the half-life estimate for position is also estimated conservatively. To determine the half-life more accurately would entail using the integral form (3.42) of the backward equation.

In the above estimates of half-life it was assumed that no control torque was used to restore the vehicle to the desired equilibrium position. It is reasonably clear that to maintain precise attitude of a space vehicle some type of control torque is needed. The example of Section 7.6 shows that a modest amount of control torque can materially increase the confinement probability. When a restoring torque is provided by an active control system, however, it is necessary also to consider the imperfections which might be present in the control system itself. A situation which may arise is typified by the recorded behavior of the Mariner IV spacecraft. A segment of the single-axis disturbance torque (computed from actual telemetry data [27]) is shown in Figure 8.2. It would appear that this disturbance torque has three components,

## FIGURE 8-2
## MARINER IV DISTURBANCE TORQUES




$\lambda = 0.01\ \mathrm{HR}^{-1}$

$\sigma$ = 4 DYNE-CM (GAS LEAKS ?)

$\mu$ = -20 DYNE-CM (REFLECTIVITY UNBALANCE?)

(a) A constant bias of about 20 dyne-cm. This bias can possibly be explained as being caused by the solar reflectivity unbalance. The total force acting on a paddle of 8000 $cm^2$ in area is about 3000 dynes; acting through a moment arm of 200 cm, this gives a torque of 600,000 dyne-cm to be balanced. Hence a 20 dyne-cm torque represents a 0.03 percent unbalance, which is probably realistic. This constant bias torque represents the major source of disturbance, but could probably be "calibrated out" in actual flight.

(b) A (Zero-mean) Poisson step process. A very crude calculation for 19 transistions indicates that (to one significant figure of accuracy) the parameters are

| | |
|---|---|
| standard deviation: | $\hat{\sigma}$ = 4 dyne/cm |
| average transition rate: | $\lambda$ = 0.02/hr |

The Jet Propulsion Laboratory engineers who have reviewed this data attribute this disturbance source to gas leaks in the control jets which were used to provide the restoring torques.

(c) A Poisson impulse process. The data reviewed was not sufficient to determine the parameters of this process, the source of which could very well be due to micrometeoroid bombardment.

If one assumes that jets which provide a smaller restoring torque would also provide proportionately small leakage torques, one can speculate that there is some optimum restoring torque level beyond which the disturbances resulting from gas leaks result in more random motion than caused by the micrometeoroids.

## 9. SUMMARY AND CONCLUSIONS

We have endeavored to show that many problems in randomly-excited dynamic systems can be studied through the partial differential equations (i.e., forward and backward equation) which govern the evolution of conditional probabilities and expectations. In particular, the stability problem is associated with the solution of the backward equation

$$\frac{\partial q(t|y)}{\partial t} = \mathcal{L}_y [q(t|y)] \tag{9.1}$$

on the interior of a region with $q \equiv 0$ on the boundary. For the process

$$dx = f(x)\, dt + G dw \tag{9.2}$$

where $dw$ is a Wiener process with covariance matrix $\Sigma dt$ the backward operator is

$$\mathcal{L}_y = f(y) \cdot \nabla_y + \frac{1}{2} \nabla_y \cdot D \nabla_y \qquad D = G \Sigma G' \tag{9.3}$$

An upper bound on $q(t|y)$ is $\hat{q}(t|y)$, the probability that the state is inside the specified region at time $t$, regardless of whether the state has left the region at some earlier time and returned subsequently; $\hat{q}(t|y)$ can be computed by integrating the transition probability density over the region, and an explicit formula for the transition probability density is available for linear systems. For nonlinear systems, the technique of calculating the confinement probability of an equivalent linear system seems promising. In section 7.6 we calculated a bound on the confinement probability for a second order linear system. Since this ad hoc procedure gave reasonable qualitative results the technique should be investigated in more detail.

The techniques for obtaining a lower bound on $q$ are not entirely satisfactory in many situations. Kushner's lower bound, based on Lyapunov theory, has the same deficiency as in deterministic systems: the quality of the estimate depends on how successful one is in guessing the appropriate V-function. Wonham's lower bound has a similar difficulty. Once a function $V(y)$ is selected to satisfy (6.3), one must find a constant $\alpha$, if such a constant exists, to satisfy (6.4). Of course, if one can find a solution to (6.4) with equality then the minimum eigenvalue of the operator gives the best lower bound in (6.5).

In many practical situations (e.g., space-vehicles) the covariance matrix $\Sigma$ is small. A readily-computed approximate solution to (9.1) would thus be quite useful, particularly if it can be used as a bound on the exact solution. One would think that the solution of (9.1) in the "noise free" case ($\Sigma = 0$) could be used to generate solutions when the noise is small. Unfortunately for asymptotically stable system (9.2) the solution to (9.1) for $\Sigma = 0$ is $q(t|y) = 1$, whereas for any $\Sigma > 0$ we showed in Section 5 that $q(t|y) \to 0$ as $t \to \infty$ so that nature of the solution has changed completely in introducing only small noise. On the other hand if one were to consider for an asymptotically stable system the first passage time to the boundary of a region given an initial state outside the region, a "low noise" solution can be obtained by proper iterations on the "noise free" solutions. If a relation could be found between this passage time and $q(t|y)$ a method of using a noise free solution to obtain an estimate of $q(t|y)$ in the low noise case would be available.

A technique suggested by W. M. Wonham should also be investigated as a means for solving the low noise problem. If we let $\alpha(t|y) = 1 - q(t|y)$ and take the Laplace transorm of (5.1) we find that

$$\begin{aligned} \mathcal{L}_y[\tilde{\alpha}(s|y)] - s\tilde{\alpha}(s|y) &= 0\,, & y \in N \\ \tilde{\alpha}(s|y) &= \frac{1}{s}\,, & y \in \partial N \end{aligned} \tag{9.4}$$

Then

$$\tilde{\alpha}(s|y) = \int_{\partial N} \tilde{\mu}(s, z)\left[\frac{\partial \tilde{p}(s, z|y)}{\partial z} \cdot \nu(z)\right] dS_z\,, \qquad y \in \partial N \tag{9.5}$$

where $\nu(z)$ is a unit outward normal to the boundary $\partial N$, $\partial\tilde{p}/\partial z$ is the gradient of $\tilde{p}$, $dS_z$ is an area element of $\partial N$, and the double layer density $\tilde{\mu}(s, z)$ satisfies

$$1 = \frac{1}{2}\,\tilde{\mu}(s\,,\,y) - \int\limits_{\partial N} \tilde{\mu}(s\,,\,z)\left[\frac{\partial \tilde{p}(s\,,\,z\,|\,y)}{\partial z} \cdot \nu(z)\right] dS_z\,, \qquad y \in \partial N \qquad (9.6)$$

For the low noise case one would attempt to solve the integral equation (9.6) by successive approximation for the first few terms. This technique deserves further investigation.

Because of the difficulties described above it appears that numerical methods, such as Monte-Carlo simulation and Galerkin's method, are likely to be the favored approach in most nonlinear systems.

White noise is taken for granted as the source of random excitation in most analytical investigations. As the above example of micrometeoroid bombardment has illustrated, this assumption may not always be justifiable. It would thus be of considerable interest to establish how much error results when white noise is used to model other processes, such as the Poisson impulse process.

In example (7.11) it was shown that when white noise is used to model the noise input to a single integrator the half-life of the system output is 32% below that obtained using a Poisson impulse model. It would be interesting to determine the class of problems for which white noise gives a conservative estimate of system stability.

Since explicit solutions to (9.1) are very difficult to obtain the problem of calculating bounds on the confinement probability was considered in Section 6. This problem deserves more attention. In example (7.1) it was found that $p_\ell(t) - p_0(t)$ was a lower bound on $q(t|0)$. Since $p_\ell - p_0$ is easy to calculate for many systems, the problem of finding general conditions for which $p_\ell - p_0$ is a lower bound should receive further attention.

Much more work is needed on the statistical characterization of the physical sources of random excitation. Even in the limited class of random excitations considered in Section 8, for which the Poisson model would appear to be reasonable, the particle flux $\varphi$ and the momentum distribution can be estimated only crudely. These sources of excitation, moreover, are probably much less significant than those caused by departure from the ideal. The solar radiation, for example, will act as a constant torque, the effect of which can far outweigh the random component of the motion. Other sources of random disturbance, such as leakage from a gas jet due to unpredictable imperfections in valve seating, are also likely to be more significant than environmental sources. If the analytical techniques of stochastic control theory are to have any practical impact, experiments will ultimately be required to obtain the data from which the statistical models can be inferred.

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27. Bouvier, H.K., "Mariner IV Disturbance Torques," JPL Interoffice Memorandum # 344-543.

## APPENDIX 1

## BIBLIOGRAPHY ON RANDOMLY-EXCITED DYNAMIC SYSTEMS AND STOCHASTIC STABILITY

### INTRODUCTION

The following is an annotated bibliography of selected books and papers dealing with various aspects of stochastic processes. No claim of completeness is made and it is hoped that this bibliography will be continuously expanded.

The bibliography is divided into 6 parts. Part 1 consists of readable introductory accounts from several viewpoints. Books and papers on the rigorous mathematical theory of stochastic processes are contained in Part 2. Various definitions and theories on stochastic stability and the related subject of renewal theory appear in Part 3. Solutions or approximate solutions to problems of engineering significance are contained in the papers of Part 4. Part 5 contains related mathematical literature. One book on deterministic stability is listed; the remaining references contain material on topics of partial differential equations which might be useful in the investigation of stochastic stability. Part 6 consists of a number of papers and reports describing physical sources of disturbances.

## I. BACKGROUND

**Barrett, J.F.,** "Application of Kolmogorov's Equations to Randomly Disturbed Automo Control Systems," Proceedings of First IFAC Congress, pp. 724-733, 1960.

**This paper** derives the forward and backward equations for the randomly perturbed dyna **system** $\dot{y} = f(y,t) + \dot{\eta}$, where $y = (y_1, y_2, \ldots, y_n)$ $\dot{\eta} = \lim (\delta\eta/\delta t)$ as $\delta t \rightarrow 0$, **subject** to the following general hypothesis: the disturbances $\delta_1\eta$, $\delta_2\eta$ in any two small **consecutive** time intervals are statistically independent. Examples of first and second-**order feedback** systems are used to illustrate the solution to special forms of the forward **equation**.

**Bharucha - Reid, A.T.,** Elements of the Theory of Markov Processes and Their Applications, McGraw-Hill Book Company, Inc., New York, 1960.

**This book** presents a nonmeasure - theoretic introduction to Markov processes. It is divided **into two parts:** Part I contains three chapters dealing successively with processes discrete **in space** and time, processes discrete in space and continuous in time, and processes **continuous** in space and time. Part II consists of six chapters devoted to applications in **biology**, physics, astronomy and astrophysics, chemistry, and operations research. Each **chapter** contains an extensive bibliography.

Chandrasekhar, S., "Stochastic Problems in Physics and Astronomy," Rev. Mod. Phys., Vol. 15, No. 1, 1943 (reprinted in N. Wax (ed.) Selected Papers on Noise and Stochastic Processes, Dover Publications, New York, N.Y., 1954.)

As its title indicates, this monograph - length paper presents many applications of the theory of Brownian motion to problems in physics and astronomy. Chapter I is devoted to the general problem of random flights and to the problem of random walk. In Chapter II the theory of Brownian motion is presented, including the Langevin equation and the Fokker-Planck Equation. Chapter III describes probability methods in the theory of coagulation, sedimentation, and the escape over potential barriers. In Chapter IV problems in stellar dynamics are considered. A rather complete set of bibliographical notes gives references to many of the original papers on Brownian motion.

Kushner, H.J., "On the Status of Optimal Control and Stability for Stochastic Systems," IEEE International Convention Record, Part 6, pp. 143-151.

This paper is a sequel to the paper presented by the author at 1965 International Convention of the IEEE (see below). The paper is concerned with linear and nonlinear diffusion models and with stochastic optimal control problems. A bibliography, consisting of 56 references in the area of stochastic control theory, is also included.

Kushner, H.J., "Some Problems and Some Recent Results in Stochastic Control," IEEE International Convention Record, Part 6, pp. 108-116, 1965.

This paper, written as a survey of recent work with some new results, presents applications of stochastic stability theory to the design of optimum feedback control systems. The paper contains some remarks on noise sources and dynamic models for random processes as well as a list of 58 references in the areas of stability and optimum control.

Lax, M., "Classical Noise III: Nonlinear Markoff Processes," Review of Modern Physics, Vol. 38, No. 2, pp. 359-379, April 1966.

This is the third paper of a series by the author in which fluctuations from the nonequilibrium steady-state are studied in detail. This paper deals with nonstationary processes and processes for which a quasilinear approximation is inadequate.

Lax, M., "Fluctuations from the Nonequilibrium Steady-State," Reviews of Modern Physics, Vol. 32, No. 25, pp. 25-63, January 1960.

This paper deals with Markovian noise in the stationary state by quasilinear methods. An extensive bibliography is included.

Lax, M. and Mengert, P., "Influence of Trapping, Diffusion, and Recombination on Carrier Concentration Fluctuations," J. Phys. Chem. Solids, Pergamon Press, Vol. 14, pp. 248-267, 1960.

In this paper the methods developed in the above reference are applied to determine properties of carrier concentration fluctuations.

Rice, S.O., "Mathematical Analysis of Random Noise," Bell System Technical Journal, Vol. 23 and 24 (reprinted in N. Wax (ed.) Selected Papers on Noise and Stochastic Processes, Dover Publications, New York, N.Y., 1954.

This monograph-length paper, devoted to the frequency domain analysis of random processes,; is divided into four parts: part I deals with the shot effect in vacuum tubes and gives expressions for the mean and standard deviation; part II consists of an analysis of power spectra and correlation functions; part III describes the statistical properties of random noise currents; and part IV considers the effect of passing noise through nonlinear devices.

Uhlenbeck, G.E. and Ornstein, L.S., "On the Theory of the Brownian Motion," Phys. Rev., Vol. 36, 1930 (reprinted in N. Wax (ed.) Selected Papers on Noise and Stochastic Processes, Dover Publications, New York, N.Y., 1954.

This paper gives the physicist's view of work (up to 1930) on the theory of Brownian motion. The general assumptions of early workers in the field are summarized and the frequency distribution of velocity and displacement are derived for a free particle in Brownian motion. The Brownian motion of a harmonically bound particle is also considered.

Wang, M.C. and Uhlenbeck, G.E., "On the Theory of the Brownian Motion II," Rev. Mod. Phys., Vol. 17, Nos. 2 and 3, 1945 (reprinted in N. Wax (ed.) Selected Papers on Noise and Stochastic Processes, Dover Publications, New York, N.Y., 1954.

This paper contains a review of the basic definitions of random processes and a comparison of two approaches to the theory of Gaussian random processes: spectrum analysis and solution to partial differential equations. It also contains an analysis of the Langevin equation and of the Brownian motion of an $n^{th}$ order linear system.

Wonham, W.M., "Advances in Nonlinear Filtering," Lecture, M.I.T., Summer Session, 1965.

These lecture notes emphasize the importance of stochastic differential equations and stochastic calculus in recent approaches to nonlinear filtering. A description of the physical significance of the Itô and Stratonovich definitions of the stochastic integral is also included.

Wonham, W.M., "Stochastic Problems in Optimal Control," IEEE Internat. Convention Record, Part 4, 1963.

This survey paper presents an introduction to stochastic control theory as it developed up to 1963. A stochastic Hamilton-Jacobi equation is used to obtain the optimum control law for a linear regulator with a quadratic performance index. A historical note and a bibliography of 53 items are also included.

## 2. MATHEMATICAL FOUNDATIONS OF STOCHASTIC PROCESSES

Doob, J.L., Stochastic Processes, Wiley, New York, 1953.

This book presents a thorough treatment of stochastic processes (both discrete parameter and continuous parameter) from a measure-theoretic point of view. Topics covered include orthogonal random variables, markov processes, martingales, processes with independent increments, processes with orthogonal increments, and linear least square prediction. A supplement on various aspects of measure theory is also included.

Dynkin, E.B., "Markov Processes and Semigroups of Operators," Theory of Probability and its Applications, Vol. 1, pp. 22-33, 1956.

In this paper the relations between various semigroups of operators and between various infinitesimal operators connected with a homogeneous in t Markov process are investigated. General conditions are established under which the Markov process is determined by its corresponding infinitesimal operator.

Let $U_t$ be a semigroup of linear operators in the Banach space L such that $\|U_t\| \leq 1$.

Let $T_t = U_t$ be an adjoint semigroup in the conjugate space $B = L^*$. More abstractly the main object of this paper can be characterized as the study of semigroups $T_t$ and its infinitesimal operators in strong and weak topologies of space B. (Author's summary)

Dynkin, E.B., "Infinitesimal Operators of Markov Processes," Theory of Probability and its Applications, Vol. 1, pp. 34-54, 1956.

This paper contains a proof of the statement: every continuous Markov process in a space of arbitrary demension is governed by

$$\frac{\partial u}{\partial t} = L u$$

where L is a generalized elliptic differential operator of second order. The paper also contains the definition of a strong Markov process, and describes the method for calculating infinitesimal operators corresponding to Markov processes.

Dynkin, E.B., Markov Processes, Academic Press, New York 1965.

This two volume book presents what has become the modern mathematical approach to the theory of Markov processes. The book is divided into five principal parts. In chapters 1-5 the general theory of homogeneous Markov processes is described using properties of infinitesimal and characteristic operators. Chapters 6-11 are devoted to additive functionals, transformations of processes and Itô's theory of stochastic integrals and stochastic integral equations. Harmonic and superhamonic functions are studied in Chapters 12 and 13. In Chapter 14 the general results are applied to the n-dimensional Wiener process. In Chapters 15-17 continuous strong Markov processes on the line are studied. An appendix on measure theory and partial differential equations is included, as well as a very complete bibliography on the mathematical theory of stochastic processes.

Khas'minskii, R.Z., "Ergodic Properties of Recurrent Diffusion Processes and Stabilization of the Solution to the Cauchy Problem for Parabolic Equations," Theory of Probability and Its Applications, Vol. 5, No. 2, 1960, pp. 179-196.

In this paper the existence of a unique invariant measure for Markov processes satisfying certain conditions is proved. This result is applied to obtain the asymptotic properties of the solution to the Cauchy problem for the parabolic equation $\partial u/\partial t = L u$ when $t \to + \infty$. It is established that these properties depend on properties of the solution to the external Dirichlet problem for the equations $L u = 0$ and $Lu = -1$. The sufficient conditions for them expressed in terms of the behavior of the coefficients in the equation $Lu = \partial u/\partial t$ are given in the appendix. (Author's summary)

Wong, E. and Zakai, M., "On the Relation Between Ordinary and Stochastic Differential Equations," Report No. 64-26 Electronics Research Laboratory, Univ. of California, Berkeley, Calif., 1964.

This paper considers the following problem: let $x_t$ be a solution (in the sense of Itô) to the stochastic differential equation, $dx_t = m(x_t, t)\,dt + \sigma(x_t, t)\,dy_t$ where $y_t$ is the Brownian motion process. Let $x_t^{(n)}$ be the solution to the ordinary differential equation which is obtained from the stochastic differential equation by replacing $y_t$ with $y_t^{(n)}$ where $y_t^{(n)}$ is a continuous piecewise linear approximation to the Brownian motion and $y_t^{(n)}$ converges to $y_t$ as $n \to \infty$. Does the sequence of solutions $x_t^{(n)}$ converge to $x_t$? It is shown that the answer is in general negative; it is however, shown that $x_t^{(n)}$ converges in the mean to the solution of another stochastic differential equation which is: $dx_t = m(x_t, t)$

$$dt + \frac{1}{2}\,\sigma(x_t, t)\left(\frac{\partial\sigma(x_t, t)}{\partial x_t}\right) dt + \sigma(x_t, t)\,dy_t .$$

Itô, K. and H.P. McKean, Jr., Diffusion Processes and Their Sample Paths, Academic Press Inc., New York, 1965.

This book presents a thorough mathematical treatment of diffusion processes: Chapters 1 and 2 describe the properties of Brownian motion. In Chapter 3 the general 1-dimensional diffusion process and its differential operators are defined. In Chapter 4 the differential generators are calculated by methods similar to those of Dynkin. Killing times and local and inverse local times are considered in Chapters 5 and 6. Chapters 7 and 8 are devoted to Brownian motion and diffusion in several dimensions. The book concludes with an extensive bibliography on diffusion processes.

## 3. STOCHASTIC STABILITY THEORY AND RENEWAL THEORY

Bershad, N. J., and P. M. DeRusso, "On the Moment Stability of Gaussian Random Linear Systems", IEEE Trans. on Auto. Contr., Vol. AC-9, No. 3; July, 1964; pp. 301-303.

In this paper upper bounds on the moments of stationary Gaussian random parameter linear systems are derived. The main result is the following theorem: Given the differential equation

$$L_q \left[ y(t) \right] + \sum_{\ell=0}^{m} \alpha_\ell(t)\, y^{(\ell)}(t) = \delta(t)$$

$$m < q, \quad y^{(\ell)}(0) = 0, \quad \ell = 0, \ \ldots, \ q - 1$$

where $L_q$ is a constant parameter qth order differential operator, and the $\alpha_\ell(t)$ are stationary, mean-zero, continuous-in-the-mean Gaussian processes, possibly correlated with each other. Then

$$\lim_{t \to \infty} \left| E\left\{ y^h(t) \right\} \right| \le c^n e^{-n} \left[ a - \frac{n}{2} c^2 S_0 \right] t$$

where

$$L_q \left[ w(t) \right] = \delta(t)$$

$$ce^{-at} = f(t) \ge \binom{k}{p} \left| w^{(k-p)}(t) \right|$$

for

$$k = 0, 1, 2, \ldots, m; \quad p = 0, 1, \ldots, k; \quad c, a, t > 0$$

$$S_o = \sum_{j=0}^{m} \sum_{k=0}^{m} \sum_{p=0}^{k} \sum_{q=0}^{j} \int_{-\infty}^{\infty} \left| E\left\{ \alpha_k^{(p)}(t)\, \alpha_j^{(q)}(t + \tau) \right\} \right|$$

Upper and lower bounds for $\lim_{t\to\infty} E\{y(t)\}$ are obtained for the system

$$L_q\, y(t) + \alpha_0(t) y(t) = \delta(t).$$

Bershad, N.J., "On the Stability of Randomly Time-Varying Linear Systems", IEEE Trans. on A/C; Vol. AC-9, No. 3; July, 1964; p. 319.

In this note sufficient conditions are stated for a random linear system to be stable according to the following definition of stability: if $H(t, v; w)$ denotes the random response of the system at time $t$ to an impulse applied at time $t - v$, then the system is stable if any only if

$$\text{Prob}\left\{ \int_0^\infty | H(t, v; w) | \, dv < c_1 < \infty \text{ for all } t \right\} = 1$$

Bertram, J. E. and Sarachik, P. E., "Stability of Circuits with Randomly Time-Varying Parameters", IRE Trans. on Circuit Theory, Vol. CT-6, pp. 260-270, 1959.

This paper gives sufficient conditions for stability, asymptotic stability and global asymptotic stability in the mean in terms of a Lyapunov function possessing certain properties. Their results state that if:

(a) A function $V(x, t)$ and its first partials in $x$ and $t$ exist and are continuous

(b) $V(0, t) = 0$

(c) $V(x, t) > \alpha \, || x ||$ for some $\alpha > 0$

then stability in the mean is guaranteed by $E\{\dot{V}(x, t)\} \leq 0$ and asymptotic stability in the mean is guaranteed by $E\{\dot{V}(x, t)\} < -g_1(||x||)$ where $g_1(0) = 0$ and $g_1(\cdot)$ is a continuous increasing function. Global asymptotic stability in the mean is guaranteed if in addition $V(x, t) \leq g_2(||x||)$ where $g_2$ is the same kind of function as $g_1$.

Bucy, R. S., "Stability and Positive Supermartingales", Jour. of Differential Equations, Vol. 1, No. 2, pp. 151-155, April 1965.

This paper presents sufficient conditions for stability in probability and for asymptotic stability almost everywhere for the nonlinear difference equation $x_n = f(x_{n-1}, r_{n-1})$. The principal results are the following theorem:

Suppose these exists a positive definite continuous function V continuous at $\infty$ such that $V(o) = 0$ and the sequence $V(\varphi(n, x_1) - x_e)$ for all $x_1 \in C(M, x_o)$ is a supermartingale with $x_e$ an equilibrium point of the system. Then $x_e$ is stable relative to $C(M, x_o)$.

Cox, D. R., Renewal Theory, John Wiley & Sons, Inc., New York, 1962.

This monograph gives a thorough treatment of renewal theory. Among the topics included are: the distribution of the number of renewals, the superposition of renewal processes, and alternating renewal processes. Applications to probabilistic models of failure and strategies of replacement are also included.

Darling, D. A. and Siegert, A. J. F., "The First Passage Time for a Continuous Markov Process", Annals of Mathematical Statistics, vol. 24, 1953, p. 624.

This paper solves the first passage problem for a strongly continuous temporally homogeneous (i.e. stationary) Markov process $X(t)$. The Laplace transform is used to obtain the distribution of $T_{ab}(x)$, a random variable giving the time of first passage of $X(t)$ from the region $a > X(t) > b$ when $a > X(o) = x > b$. From the distribution of $T_{ab}$ the distribution of the maximum of $X(t)$ and the range of $X(t)$ are calculated. Certain statistical problems in sequential analysis, nonparametric theory of "goodness of fit," and optional stopping are included to illustrate the results.

Gersch, W., "The Solution, Stability and Physical Realizability of the First Order Linear System with White Noise Parameter Variations," 1966 JACC Proceedings, pp. 435-440.

This paper is concerned with the interpretation of white noise parameter first-order differential equations. It is indicated that a deterministically unstable first-order system can be stabilized with a white noise parameter variation. Various forms of stochastic stability definitions are examined with reference to the first-order system.

Khas'minskii, R.Z., "On the Stability of the Trajectory of Markov Processes," PMM Vol. 26, pp. 1025-1032, 1962.

In this paper necessary and sufficient conditions are established for stability with probability one of the origin of the n-dimensional system

$$\dot{x} = b(x, t) + \sigma(x, t)\dot{\xi}$$

where $\dot{\xi}$ is n-dimensional white noise and $\sigma$ is an n x n matrix with components $a_{ij}$. The principal result is the following theorem: If (1) $a_{ij}(x) = 0(|x|^2)$, $b_i(x) = 0(|x|)$, $x \to 0$ and (2) some continuous function $m(x)$, which is positive when $x \neq 0$, and for all real $\lambda_i$ the inequality

$$\sum_{i,j=1}^{n} a_{ij}\lambda_i\lambda_j \geq m(x) \sum_{i=1}^{n} \lambda_i^2$$

is valid then a necessary and sufficient condition for stability with probability 1 is the existence, in some neighborhood of $x = 0$, of a continuous non-negative function $V(x)$ which vanishes only at $x = 0$, and for which $\mathcal{L}\, V(x) \le 0$, where

$$\mathcal{L}(\quad) \equiv \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial^2(\quad)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t, x) \frac{\partial(\quad)}{\partial x_i}$$

This result is similar to that of Kushner. However, restriction (1) is explicitly mentioned in this paper whereas in Kushners' paper it becomes evident only upon examining the examples.

Kozin, F., "On Almost Sure Stability of Linear Systems with Random Coefficients", Jour. of Math. Physics, Vol. 43, pp. 59-67, 1963.

This paper presents a sufficient condition for almost sure asymptotic stability in the large for linear systems with continuous, stationary, ergodic coefficient processes. The main result is the following theorem:

If the solution of the constant coefficient system $\dot{x} = Ax$ is asymptotically stable in the large, and if the non-identically zero elements $\{f_{ij}(t);\ t \in [0, \infty)\}$ of the matrix $F(t)$ of the system $\dot{x} = [A + F(t)]\, x$ are stochastic processes which satisfy certain other conditions, then there is a constant $a' > 0$ depending upon the matrix $A$ such that $E\{\|F(t)\|\} < a'$ implies that the solution of $\dot{x} = [A + F(t)]\, x$ is almost surely asymptotically stable in the large.

An example is included of a second-order system where the constant coefficient terms constitute a damped oscillating system.

Kozin, F., "On Relations Between Moment Properties and Almost Sure Lyapunov Stability for Linear Stochastic Systems", Jour. of Math. Anal. & Appl., Vol. 10, pp.342-353, 1965.

This paper presents a sufficient condition for almost sure asymptotic Lyapunov stability in the large. The major result is the following theorm:

If for $x_o \in R_n$, $t_o > 0$, the solution process of the system

$$\dot{x} = [A + B(t)]\, x$$

(where $A$ is a constant $n \times n$ matrix and $B(t)$ is an $n \times n$ matrix whose non-identically zero elements are stochastic processes) satisfies

$$\int_{t_o}^{\infty} E\left\{ \| x(t; x_o, t_o) \| \right\} dt < \infty$$

then the stationary solution $x_1 = x_2 = \ldots = x_n = 0$ is almost surely asymptotically stable in the Lyapunov sense in the large relative to $R_n$.

The paper also gives two examples investigating almost sure asymptotic stability and stability in the mean for the system $\dot{x} + f(t)\, x = 0$ where $f(t)$ is a sample function from a random process.

Kushner, H. J., "On The Construction of Stochastic Lyapunov Functions", to appear in IEEE Trans. GAC

This paper presents a stochastic analog of the deterministic method of partial integration for computing Lyapunov functions. Examples are used to illustrate the method.

Kushner, H. J., "Finite Time Stochastic Stability and the Analysis of Tracking Systems".

This paper gives a method for obtaining bounds to the quantities $P_x\left\{ \sup_{0 \le t \le T} \|x_t\| \ge \epsilon \right\}$, $P_x\left\{ \sup_{0 \le n \le N} \|x_n\| \ge \epsilon \right\}$, $P_x\left\{ \sup_{0 \le t \le T} V(x_t) \ge \lambda \right\}$, and $P_x\left\{ \sup_{0 \le t \le T} V(x_n) \ge \lambda \right\}$ where the stochastic processes $x_t$, $t \le T$, or $x_n$, $n < N$, are components of Markov processes (continuous or discrete parameter, respectively); $P_x(B)$ is the probability of the event $B$ given that the initial state $x_o$ equals $x$; and $V(x)$ is a continuous non-negative function.

The method of obtaining estimates, presented in a number of theorems and corollaries, involves finding stochastic Lyapunov functions. Thus the method suffers from some of the practical deficiencies of the Lyapunov function approach to the stability of ordinary differential equations. Examples are included to illustrate the method.

Kushner, H. J., "Stochastic Stability and the Design of Feedback Controls", Tech. Report 65-5, Center for Dynamical Systems, Brown University, May 1965.

This report is concerned with stochastic extensions of various techniques for using the second method of Lyapunov to aid the construction and analysis of the feedback control system $dx = f(x, u)dt + \sigma(x, u)\, dz$. To each control $u$ there is the associated cost $C^u(x) = E_x^u \int_0^{\tau_u} k(x_t, u_t)dt$ where $\tau_u$ is the random time of arrival at $\partial S$, the boundary of a set S, and $k(x, u)$ is continuous and non-negative. The object of the control system is to transfer $x = x_o$ to $\partial S$ in finite average time and minimize $C^u(x)$. Theorems are proved comparing min $C^u(\overset{o}{x})$ with certain non-negative, scalar valued functions $V(x)$. Examples are presented to illustrate application of the theorems to the problem of choosing and analyzing the effect of feedback controls for several stochastic systems.

Kushner, H. J., "On the Stability of Stochastic Dynamical Systems", Proc of N. A. S., Vol. 53, pp. 8-12, 1965.

This paper presents several results concerning a Lyapunov function approach to the stability of stochastic dynamical systems. Theorems on stability with probability one and on asymptotic stability with probability one are proved for the discrete-time system $x_{n+1} = f(x_n, y_n)$ where $y_n$ is a Markov process.

Kushner, H. J., "On the Theory of Stochastic Stability", Tech. Report 65-1, Center for Dynamical Systems, Brown University, January 1965.

This report establishes a number of theorems on stability with probility one and on asymptotic stability with probability one for the continuous-time system $dx = f(x)\,dt + \sigma(x)\,dz$ where z is vector Brownian motion with independent components. Examples are included to illustrate application of the theorems for linear and nonlinear systems.

Samuels, J. C., "On the Stability of Random System", J. Acoust. Soc.; Vol. 32; pp. 594-661; May, 1960.

This paper develops a theory for the analysis of a linear system in which many parameters vary as white noise stochastic processes. Mean square stability criteria are obtained for such systems and an example of an RLC circuit with both resistance and capacitance random variations is included.

Samuels, J. C., "On the Mean Square Stability of Random Linear Systems", IRE Trans. on Circuit Theory, Vol. CT-6, pp. 248-259; May, 1959.

This paper develops the mean square stability theory of Samuels and Ernigen and extends it to systems containing one or more stochastic parameters which may be dependent. An example of an RLC circuit with a narrow-band capacitance variation is considered.

Samuels, J. C., and A. C. Ernigen, "On Stochastic Linear Systems", J. Math. Phys., Vol. 38; pp. 83-103; July, 1959.

This paper is concerned with an analysis of $n^{th}$ order linear systems with random coefficients. Three types of systems are described: systems with small randomly varying coefficients, systems with slowly varying random coefficients, and systems containing only one random coefficient. For the latter class the authors define "mean square stability" in terms of the second moment of the position component of the state vector and analyze the stability of an RLC circuit and a bar under a random axial load.

Siegert, A. J. F., "On the First Passage Time Probability Problem", Physical Review, Vol. 81, No. 4 pp 617-623.

Using an integral equation, this paper derives an exact solution for the first passage time probability of a stationary one-dimensional Markov process. A recursion formula for the moments of the first passage time probability is given for the case where the conditional density satisfies a Fokker-Planck equation. The Wiener-Rice series for the recurrence time probability density is derived for a two-dimensional Markov process.

Wang, P. K. C., "On the Almost Sure Stability of Linear Stochastic Distributed-Parameter Dynamical Systems", presented at ASME Winter Annual Meeting, 7-11 November 1965.

This paper presents sufficient conditions for almost sure stability and asymptotic stability of linear stochastic distributed-parameter dynamical systems described by linear partial differential or differential-integral equations with stochastic parameters. Examples are given to illustrate the results.

Wonham, W. M., "On First Passage Times For Randomly Perturbed Linear Systems" (in preparation).

This paper gives a method for estimating the probability of the first passage time to the boundary of a domain D. The principal result is the following theorm:

Let $V(x)$ be a twice continuous differentiable function such that

$$V(x) = 0, \quad x \in \dot{D}$$

and

$$0 \leq V(x) \leq 1, \quad x \in D$$

where D is an open connected domain D with smooth boundary $\dot{D}$, such that D contains the origin $x = 0$. If there exists a constant $\lambda > 0$ such that

$$\mathcal{L}[V(x)] + \lambda V(x) \geq 0, \quad x \in D,$$

where $\mathcal{L}$ is the differential operator for the system, then

$$P(t, x) \geq V(x)e^{-\lambda t}, \quad t > 0$$

where

$$P(t, x) = P\left\{ x(s) \in D, \quad 0 \leq s < t \mid x(0) = x \right\}$$

When $\mathcal{L} + \lambda$ is self-adjoint an upper bound on the principal eigenvalue $\lambda_1$ can be obtained by Dirichlet's principle. When $\mathcal{L} + \lambda$ is not self-adjoint there appears to be no straightforward means for obtaining $\lambda_1$. Examples for a number of linear systems are used to illustrate the procedure.

Wonham, W. M., "A Lyapunov Method for the Estimation of Statistical Averages", Tech. Report 65-6, Center for Dynamical Systems, Brown University, June 1965.

In this paper a Lyapunov criterion is obtained for the existence of the stationary average $E\left\{ L(x) \right\}$ for the system

$$\dot{x} = f(x) + G(x) \ \xi(t), \quad t > 0$$

where $x$, $f$ are n-vectors, $G$ is an $n \times n$ matrix, $\xi(t)$ is n-dimensional Gaussian white noise, and $L$ is an arbitrary nonnegative function. A method for calculating an upper bound for $E\left\{ L(x) \right\}$ is also presented. The results are applied to an example where the unperturbed system $\dot{x} = f(x)$ is of Lur'e type.

## 4. SPECIFIC PROBLEM SOLUTIONS

Barrett, J. F., "Application of Kolmogorov's Equations to Randomly Disturbed Automatic Control Systems," Proceedings of First IFAC Congress, pp. 724-733, 1960.

This paper gives the steady state solution to the forward equation for the first and second-order systems

$$\text{(a)}\quad \dot{y} = -f(y) + \xi,$$
$$\text{(b)}\quad \ddot{y} + a\dot{y} + f(y) = \xi$$
$$\text{(c)}\quad \dot{y} = \frac{1}{T^2}\,\mathrm{sgn}\,(e - ky)$$
$$\dot{e} = -y + \xi$$

and for the $n^{th}$ order system

$$\text{(d)}\quad \dot{y} = Ay + \xi$$

where f(y) in (a) and (b) has a saturating characteristic, A in (d) is a constant nxn matrix, and $\xi$ is Gaussian stationary white noise.

Bergen, A.R., "Random Linear Systems: A Special Case," Trans. A. IEE, Vol. 80 (Applications and Industry), pp. 142-145; July, 1961.

In this paper frequency domain techniques are used to calculate the mean-square error for a linear control system with a single random gain element.

Blachman, N.M., "On the Effect of Noise in a Non-linear Control System," Proc. First IFAC Conference, Moscow (1960), Vol. 1., pp. 810-815.

This paper considers the effect of weak white noise on a weakly-damped nonlinear oscillatory system. When the energy of the system reaches a certain point unstable oscillations occur. The problem is to determine the probability of transistion of the system through the boundary of instability. The steady-state solution to the problem is obtained.

Gray, A.H., "First-Passage Time in a Random Vibrational System," presented at ASME Winter Annual Meeting, 7-11 November 1965.

In this paper the first passage-time problem for a random second-order vibrational system is solved by means of an approximation which converts a two-dimensional Markov process to a one-dimensional Markov process. Laplace transforms are used to evaluate the mean square time to failure. The problem arises in determining structural response to earthquakes.

Kushner, H.J., "Finite Time Stochastic Stability and the Analysis of Tracking Systems."

This paper gives a number of examples illustrating calculation of a bound on the probability that a simple tracking system, will exceed its maximum tracking error. The method involves finding stochastic Lyapunov functions and applying certain theorems developed in the paper.

Ruina, J.P. and Van Valkenburg , M.E., "Stochastic Analysis of Automatic Tracking Systems," Proc. First IFAC Conference, Moscow (1960), Vol. 1., pp. 810-815.

This paper presents an analysis of automatic tracking systems using the Fokker-Planck equation to find the probability density of the output of the tracking system. A bound on the probability of losing track is found for the system

$$\dot{x} = - ax + \xi (t)$$

where $\xi$ (t) is white Gaussian noise. In an interesting discussion to this paper R.L. Stratonovich describes a method for obtaining an exact solution. The Stratonovich approach uses the method of reflections, similar to the method used in electrostatics.

Wonham, W.M., "On the Probability Density of the Output of a Low-Pass System When the Input is a Markov Step Process," Trans. IRE PGIT. IT-6 (5), 1960, pp. 539-544.

In this paper forward equations are derived for the (n + 1) - dimensional Markov process generated when a Markov step process $\{ s (t) \}$ is the input to an $n^{th}$ order system, $\dot{x} = f (x ,s)$. An interesting analogy is drawn between the forward equation and Boltzmann's equation in the kinetic theory of gases. As examples, a symmetric three-level signal smoothed by an RC low-pass filter and a doubly integrated telegraph signal are considered.

Wonham, W.M., "Stochastic Analysis of a Class of Nonlinear Control Systems with Random Step Inputs," JACC Preprints, 1962.

This paper considers the control system

$$\dot{Y} = u \; [X(t), \; Y ]$$

where Y(t) is the output at time t and the input (desired output) $\{ X (t) \}$ is a Markov step process. The forward equation for the joint process $\{ X(t) , Y (t) \}$ is derived and solved for a relay system in which $\{ X (t) \}$ is a simple jump process.

## 5. RELATED MATHEMATICAL LITERATURE

### (a) Deterministic Stability

LaSalle, J.P. and Lefschetz, S., Stability by Lyapunov's Direct Method, Academic Press, New York, 1961.

This book contains a description of Lyapunov's second method for deterministic systems, including treatment of Lagrange Stability, Ultimate Boundedness, and the problem of Lu're.

### (b) Partial Differential Equations

Freidlin, M.I., "The Dirichlet Problem for an Equation Involving a Small Parameter and with Discontinuous Coefficients," Soviet Math. Dokl., Vol. 3, pp. 767-770, 1962.

This paper considers the following problem:

$$L^\alpha u^\alpha(x) = \alpha^2\left(\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u^\alpha}{\partial x_i\,\partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u^\alpha}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u^\alpha}{\partial x_i} = 0,\ x \in D-S$$

$$\lim_{x \to x_0} u^\alpha(x) = \psi(x_0),\ x_0 \in \Gamma$$

$u^\alpha(x)$ and grad $u^\alpha(x)$ are continuous for $x \in D$

where D is a given domain with smooth boundary $\Gamma$ in an n-dimensional Euclidean space $R^n$, S is an (n-1) dimensional manifold $S \subset D$, and the above elliptic operator is nondegenerate in $D \cup \Gamma$.

Sufficient conditions are established for the existence of the limiting solution $u(x) = \lim_{\alpha \to 0} u^\alpha(x)$.

Freidlin, M.I., "A Mixed Boundary Value Problem For Elliptic Differential Equations of Second Order With a Small Parameter," Soviet Math. Dokl., Vol. 3, pp. 616-620, 1962.

This paper considers the following problem:

$$L^\alpha u^\alpha(x) = \frac{\alpha^2}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u^\alpha}{\partial x_i\,\partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u^\alpha}{\partial x_i} = 0,\ x \in D$$

$$u^\alpha(x)\big|_{x\in\Gamma_1} = \psi(x)$$

$$\frac{\partial u^\alpha}{\alpha \ell}\bigg|_{x\in\Gamma_2} = 0$$

where D is a given domain with boundary $\Gamma$ in an n-dimensional Euclidean space $R^n$, $\Gamma_2$ is a subset of $\Gamma$ open in $\Gamma$, $\Gamma_1 = \Gamma - \Gamma_2$, $\psi(x)$ is a continuous function on $\Gamma_1$, $\ell(x)$ is a field of functions whose third derivatives are continuous, and the above elliptic operator is nondegenerate on $D \cup \Gamma$.

Sufficient conditions are established for the existence of the limiting solution $u(x) = \lim_{\alpha \to 0} u^{\alpha}(x)$.

Sobolev, S.L., Partial Differential Equations of Mathematical Physics, Pergamon Press, 1964.

This book is based on a coruse of lectures given in the State University in Moscow and the chapters are arranged into thirty "lectures". Topics covered include the theory of integral equations, Green's function, and Lebesque integration. Examples from heat condition, hydrodynamics, and the theory of sound are used to introduce and illustrate the mathematical techniques.

# 6. PHYSICAL SOURCES OF DISTURBANCES

———— "The Natural Environment for the MOL System Program", AFCRL-64-845, (AD 455 556), 25 October 1964.

This report discusses in some detail the space environment between the altitudes of 100 km and 700 km above the Earth's surface. The following are examined in some detail with some estimates of their probabilistic occurrences.

1. Energetic particle radiation
2. Solar Flares
3. Meteoroids
4. Solar Electromagnetic Emissions
5. Galactic Cosmic Rays

Alexander, W.M. "Cosmic Dust", Science, Vol. 138, No. 3545, Dec. 7, 1962, pp 1098-1099

The cosmic dust experiment of Mariner II is described and an estimate of the flux of dust necessary for .9 probability of at least one impact for the time of measurement is given as $6 \times 10^{-6}$ particles/$m^2$-sec. Assuming an average dust velocity as 55 km/sec, the mass of the dust particles was estimated to be $1.3 \pm 0.3 \times 10^{-10}$ gms

Briggs, R.E., "Steady State Space Distribution of Meteoric Particles under the Operation of the Poynting-Robertson Effect", The Astronomical Journal, Vol. 67, No. 10, Dec. 1962, pp. 710-723.

A model is proposed for the distribution of meteoric particles in interplanetary space The Poynting-Robertson effect is applied to derive the steady state distribution of orbits, and to compute the space density of particles at many points in the solar system. The space density $n_s(\rho, z)$ of particles having radii greater than s is

$$n_s(\rho, z) = \frac{H(\rho, z)}{C_3} \int_s^{\infty} C\left(\frac{3}{\rho^{k-1} s^{3k}} + \frac{\rho'}{\rho^{k} s^{3k-1}}\right) ds$$

where $\rho$ = radial distance from the sun in the plane of the ecliptic
z = distance normal to the ecliptic

Dalton, C.C., "Cislunar Meteoroid Impact and Puncture Models with Predicted Pegasus Satellite Punctures", NASA TM X-53187, 13 Jan. 1965.

In cislunar space at a distance of h km from the earth's surface, $(10^2 \leq h \leq 10^6)$, the model for the mean meteoroid flux $F_s$ (meteoroids/sec-$m^2$) of a mass equal to or greater than M grams $(10^{-10} \leq M \leq 1)$ impacting a randomly oriented orbiting vehicle is predicted to be

$$\log F_s = 14.92 \pm 0.06 + \beta_2 \log M$$

where $\beta_2 = -1 - 0\ 34\ e^{0\ 26(\log h - 2)} + 0\ 24(\log h - 2)$

Using the above, and a model for the puncture flux, and an estimated mean meteoroid density of 0.44g/cc and a velocity of 26 7 km/sec a flux distribution is predicted for a Pegasus satellite

Evans, J W , (ed ), The Solar Corona, Academic Press, New York, 1963

This text describes the basic solar flare event and all flare associated phenomena No statistics concerning the occurrence of these events are set forth nor are any mechanisms proposed which might yield a better understanding of the solar-flare event

McDonald, F B , (ed.), "Solar Proton Manual", NASA TR-R169, December 1963

This report highlights the sun's activity from 1960-1962 with detailed correlation between sunspots (solar flares) and particle flux at various earth locations. Due to the sun's rotation, there is a preferred orientation of the solar flux impingement on the earth No information is given as to the estimated original flux or its decay in space or time

Miroshnichenko, L I , "Cosmic Rays from the Sun", Piroda (Nature), Number 1, Moscow, Jan 1965, pp 25-34 (Translated as N65-20625 or N65-18183)

This report studies Solar Corpuscular Radiation in the high energy class (with energy greater than 1MEV(nucleon). These particles which consist of protons and heavier nuclei in the $1 - 10^5$ MEV range are emitted by the sun during chromosphere bursts and closely resemble cosmic rays The energey density reaches a maximum of $10^{-9}$ ergs/cm$^3$ During one period of high solar activity a solar flux with energies of 49 - 500 MEV reached an intensity of $1\ 5 \times 10^7$ particles/cm$^2$ - sec was observed.

Rohrbach, E J., and Goldstein, H S , "The Space Radiation Environment", Machine Design, Vol 34, No 25, Oct 1962, pp 146-150

Discussion of the two types of space radiation, namely, solar and galactic, are aescribed in terms of the number of particles with given energy or less per unit time per unit area Poisson distributions are taken to be characteristic of both types of radiation

Whipple, F L "On Meteoroids and Penetration", J of the Astronautical Sciences, Vol 10, No. 3, Fall 1963, pp 92-94

By reviewing photographic meteor data the following values are obtained for a zero magnitude (visual) meteor.

| | |
|---|---|
| meteoroid density. | 0.44 gr/cc |
| velocity | 30 km/sec |

## APPENDIX 2

## MONTE CARLO SIMULATION PROGRAM

### Purpose

To determine the lifetime of a process governed by the stochastic differential equations

$$\dot{x} = f(x, \xi, t) \qquad x(t_0) = x_0 \qquad (1)$$

where $x$ is the state vector dim $\leq$ 15

$t$ is the time

$\xi$ is a vector of random noise, gaussian distributed with zero mean and standard deviation $\sigma$

### Method

Equation (1) is integrated starting at time $t_0$ until either ($i$) the time $t$ exceeds some limit T or

(ii) until $|x_i|$ exceeds some limit $B_i$

The integration step number (integration step size times no. of steps) at which this occurs is recorded. An ensemble of trajectories is run and the number of trajectories leaving at each step is recorded.

### Usage

The program requires that the user supply a subroutine DERIV, to compute the derivatives

i.e., DX(I) = F(X(J), RV(J), T)

DX = derivative $x_i$

X = state vector

RV = random variable vector, gaussian distributed,

T = time

The calling statement is

CALL DERIV (DX, X, RV, T)

Additional input data are stored in COMMON - COMMON BLC(6), IB(8), H, $T_0$, RN, FC(5) in data blocks IB and FC.

Output

Program prints first trajectory and tabulates no. of trajectories leaving at each step.

Input

| Card # | Format | Description | Column |
|---|---|---|---|
| 1 | (I6) | Run No. | 1-6 |
| | (11A6) | Alphameric Description | 7-72 |
| 2 | (I5) | No. of steps, at step size H | 1-5 |
| | | No. of ensemble members | 6-10 |
| | | Frequency of print-out | 11-15 |
| | | Fixed point random number | 16-20 |
| | | | 21-25 |
| | | Dimension of state | 26-30 |
| | | Input Vector IB(I) | 31-70 |
| 3 | (7F10.5) | Step Size | 1-10 |
| | | Initial time $T_0$ | 11-20 |
| | | Initial random number | 21-30 |
| | | Input vector FC(I) | 31-70 |
| 4 | (7F10.5) | $X_0$ | |
| 5 | (7F10.5) | Standard deviation | |
| 6 | (7F10.5) | $B_\ell$ | |

```
$IBFTC MARKOV  NODECK
C
C     ****************************************************************
C
C         PURPOSE
C         MARKOV COMPUTES THE PROBABILITY THAT STATE OF A MARKOV PROCESS
C         DESCRIBED BY THE VECTOR DIFFERENTIAL EQUATION
C
C                 .DXS/DT= F( XS, RV, T )
C         WHERE
                       RV IS RANDOM NOISE WITH STANDARD DEVIATION VARN
C                      DXS   = DERIVATIVE OF THE STATE XS
C                      XS    =   STATE
C         WILL EXIT FROM AN N DIMENSIONAL BOX DESCRIBED BY THE VECTOR BX
C
C         SUBROUTINES    REQUIRED
             DERIV USER PROVIDED TO COMPUTE THE DERIVATIVES
C            JINPG
C            RANDG
C
C                INPUT  DATA
C                      NMAX  =  NO OF STEPS
C                      MSMB  =  NO OF ENSEMBLE  MEMBERS
C                      MPRN  =  FREQUENCY  OF PRINT-OUT. NO OF STEPS
C                               BETWEEN  EACH PRINT
C                      IRAND =  FIXED POINT RANDOM NO.
C                      F1    =  RANDOM  FLOATING  PT NO.
                       I2 DIMENSION OF STHE STATE VECTOR
C                      XZERO    =  STARTING STAE
C                      BX    =  DIMENSIONS OF BOX
C                      VARN  =  STANDARD  DEVIATION OF  NOISE
C
C     ****************************************************************
C
      COMMON NMAX,MSMB, MPRN, IRAND, I1,I2,I3,I4,I5,I6,I7,I8,I9,I10
      COMMON XO,B, F1,F2,F3,F4,F5
      DIMENSION  XS(15), DXS(15), XZERO(15), RV(15),VARN(15), BX(15)
      DIMENSION   ALP(11),IRE(5000) , G(8) , ISTEP(20)
C
C         READ INPUT DATA AND INITAILIZE PROGRAM
C
    1 READ (5,901)  INO, (ALP(I),I=1,11)
  901 FORMAT(I6, 11A6 )
      READ (5,902)   NMAX , MSMB, MPRN, IRAND,  I1, I2, I3, I4, I5,I6,
     1I7, I8, I9, I10
  902 FORMAT( 14I5)
      READ (5,903)   XO,  B, F1,F2,F3,F4,F5
  903 FORMAT(7 F10.5)
      READ (5,700)   (XZERO(I),I=1, I2 )
      READ (5,700)   (VARN(I), I=1, I2 )
      READ (5,700)   (BX(I), I=1, I2   )
```

```
  700 FORMAT(7 E10.4)
      WRITE (6,800) INO, (ALP(I), I=1,11 )
  800 FORMAT(1H1, 20X, 30H RANDOM PROCESS SIMULATION        /
     110X, 10H RUN NO.    I5, 10X, 11A6//)
      IPR1 = 2*I1+1
      IPR2=IPR1+1
      RN  =F1
C               INITIALIZE  RANDOM  NUMBER GENERATOR AND SUMMATION
      DO  10 I=1, IRAND
   10 RN1=RANDG(RN)
      WRITE(6,802) NMAX,MSMB
  802 FORMAT(10X,13H NO.OF STEPS  I5, 10X, 27H NO. OF MEMBERS OF ENSEMBL
     1E  I5/ )
      WRITE(6,701)  (XZERO(I),I=1, I2 )
      WRITE(6,702)  (VARN (I),I=1, I2 )
      WRITE(6,703)  (BX(I),I=1,I2)
  701 FORMAT(1H / 10X, 15H INITIAL STATE   /  3(5E20.8 / )   )
  702 FORMAT(1H / 10X, 15H STD DEV         /  3(5E20.8 / )   )
  703 FORMAT(1H / 10X, 15H BOUND           /  3(5E20.8 / )   )
C               COMPUTE  THE STATE
      TLIM = B+ FLOAT(NMAX)*XO
      DO  12    I =  1,NMAX
   12 IRE(I)=0
C
C              INITIALIZE   FOR  START  OF  ENSEMBLE
C
   15 DO  30    IEN=1,MSMB
      N=1
      T= B
      ISW = -1
      DO 16  I=1,I2
   16 XS(I) = XZERO(I)
   20 CONTINUE
   18 DO  17 I=1,I2
   17 RV(I) =  VARN(I)* RANDG(RN)
   19 CALL DERIV(DXS, XS, RV,  T)
      CALL  JINPG( XS,DXS, T, TLIM, XO, ISW, I2,EMAX )
      IF( ISW. EQ. 0 ) GO TO 18
      IF( IEN. NE. 1) GO TO  22
      WRITE(6,704)  N, T, (XS(I), I=1, I2)
  704 FORMAT(1H / 10X, 10H STEP NO.   I5, 5X, 6H TIME  F10.5/
     1 3( 10X, 5E20.8/   )       )
   22 N= N+1
      DO  23 I=1,I2
   23 IF( ABS(XS(I)). GT.  BX(I) ) GO TO 25
      IF( N. LE. NMAX) GO TO 20
      GO TO 30
   25 IRE(N-1)  = IRE(N-1) +1
   30 CONTINUE
      WRITE (6,800) INO, (ALP(I), I=1,11 )
      WRITE(6,802) NMAX,MSMB
C
```

```
C              ACCUMULATE  AND  PRINT   SUM
C              PRINOUT  IS  10  POINTS PER LINE
C
      DO   100    I=1,NMAX
  100 IRE(I+1)  =  IRE(I+1) +IRE(I)
      M10   =MPRN*10
      NULIN =  NMAX /M10
      IF(  MOD(NMAX, M10). EQ. 0)  GO TO 110
      NULIN =   NULIN+1
  110 DO  150   I=1, NULIN
      L1 = M10*(I-1) + MPRN
      L10  =   L1+  9*MPRN
      IF( L10. LE. NMAX) GO TO 115
      L10  = NMAX
  115 ISTEP(1) =L1
      DO  120  J=2, 10
  120 ISTEP(J) = ISTEP(J-1) + MPRN
      WRITE (6, 940)  (ISTEP(J), J=1,10)
      WRITE (6, 941)  (IRE(J),  J=L1,L10, MPRN       )
  150 CONTINUE
  940 FORMAT(1H // 5X, 15H STEP NUMBER     , 10X,  10I10 )
  941 FORMAT(5X, 22H NO. OF MEMBERS.GT.B     , 3X, 10 I10)
      GO TO 1
      END
```

```
$IBFTC DERIV    NODECK
      SUBROUTINE  DERIV(DY,Y, RV, T)
C
C        SAMPLE SUBROUTINE DERIVATIVE
C
C     FUEL-OPTIMUM SINGLE-AXIS ATTITUDE CONTROL
C
      REAL K
      DIMENSION DY(3),Y(3),RV(2)
      COMMON I(14),F(3),F2,F3,F4,F5
      K = F2
      A = F3
C     GENERATION OF FEEDBACK CONTROL
      S = Y(1) + K*Y(2)
      IF(S) 10,10,11
   10 IF(Y(2)) 1,2,2
    1  U = 1.0
       GO TO 50
    2  U = 0.0
       GO TO 50
   11 IF(Y(2)) 2,2,3
    3  U =-1.0
       GO TO 50
C     EQUATIONS  OF  MOTION
   50 DY(1) =  A*U  +  Y(3)  +RV(1)
      DY(2) =  Y(1)
      DY(3) =  RV(2)
      RETURN
      END
$ENTRY MARKOV  MARKOV
$IBSYS
```

SAMPLE OUTPUT

```
              RANDOM PROCESS SIMULATION
     RUN NO.  60166                      SINGLE AXIS ATTITUTDE CONTROL    A =.05  K=1


     NO.OF STEPS   240          NC. OF MEMBERS OF ENSEMBLE   20


     INITIAL STATE
0.                   -0.

     STD DEV
0.10000000E 01        0.

     BOUND
0.99999998E 07        0.10000000E 01

     STEP NC.     1       TIME    0.10000
        -0.12842491E-01       0.80063278E-03

     STEP NC.     2       TIME    0.20000
        -0.82720243E-01      -0.47589776E-02

     STEP NC.     3       TIME    0.30000
        -0.67890835E-01      -0.13476893E-01

     STEP NC.     4       TIME    0.40000
        -0.11039052E 00      -0.21807075E-01

     STEP NC.     5       TIME    0.50000
        -0.10097639E 00      -0.33345081E-01

     STEP NC.     6       TIME    0.60000
        -0.15027185E 00      -0.43919767E-01

     STEP NC.     7       TIME    0.70000
        -0.16595329E 00      -0.58916400E-01

     STEP NC.     8       TIME    0.80000
        -0.10820085E 00      -0.72440274E-01

     STEP NC.     9       TIME    0.90000
        -0.12324748E 00      -0.83176114E-01

     STEP NC.    10       TIME    1.00000
        -0.10095365E 00      -0.95938575E-01

     STEP NC.    11       TIME    1.10000
        -0.65556996E-01      -0.10467781E 00

     STEP NC.    12       TIME    1.20000
         0.47335068E-01      -0.10723718E 00

     STEP NC.    13       TIME    1.30000
         0.70172192E-01      -0.10067324E 00

     STEP NC.    14       TIME    1.40000
         0.65814529E-01      -0.93353354E-01

     STEP NC.    15       TIME    1.50000
         0.31558616E-01      -0.87127298E-01

     STEP NC.    16       TIME    1.60000
         0.86886139E-01      -0.79095487E-01
                          ...
```

RANDOM PROCESS SIMULATION
RUN NO. 60166 SINGLE AXIS ATTITUDE CONTROL A =.05 K=1

NO.OF STEPS 240 NO. OF MEMBERS OF ENSEMBLE 20

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| STEP NUMBER | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| NO. OF MEMBERS.GT.B | 0 | 0 | C | 0 | 0 | 0 | 0 | C | 0 | 0 |
| STEP NUMBER | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| NO. OF MEMBERS.GT.B | 0 | 0 | C | 0 | 0 | 0 | 0 | C | 0 | 0 |
| STEP NUMBER | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| NO. OF MEMBERS.GT.B | 0 | 0 | C | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| STEP NUMBER | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| NO. OF MEMBERS.GT.B | 0 | 0 | C | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| STEP NUMBER | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| NO. OF MEMBERS.GT.B | 1 | 1 | 1 | 1 | 1 | 2 | 3 | 3 | 3 | 3 |
| STEP NUMBER | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| NO. OF MEMBERS.GT.B | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| STEP NUMBER | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| NO. OF MEMBERS.GT.B | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 |
| STEP NUMBER | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| NO. OF MEMBERS.GT.B | 4 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| STEP NUMBER | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| NO. OF MEMBERS.GT.B | 5 | 5 | 6 | 6 | 6 | 6 | 7 | 7 | 7 | 7 |
| STEP NUMBER | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| NO. OF MEMBERS.GT.B | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| STEP NUMBER | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 | 109 | 110 |
| NO. OF MEMBERS.GT.B | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| STEP NUMBER | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 |
| NO. OF MEMBERS.GT.B | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 8 | 8 |
| STEP NUMBER | 121 | 122 | 123 | 124 | 125 | 126 | 127 | 128 | 129 | 130 |
| NO. OF MEMBERS.GT.B | 8 | 8 | 8 | 8 | 10 | 10 | 10 | 11 | 11 | 11 |
| STEP NUMBER | 131 | 132 | 133 | 134 | 135 | 136 | 137 | 138 | 139 | 140 |
| NO. OF MEMBERS.GT.B | 12 | 12 | 12 | 12 | 12 | 12 | 13 | 13 | 13 | 13 |



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